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Ensembles minimaux, existence et régularité

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Résumé

Cette thèse s'intéresse principalement à l'existence et à la régularité des ensembles minimaux.

On commence par montrer, dans le chapitre 3, que le problème de Plateau étudié par Reifenberg admet au moins une solution. C'est-à-dire que, si l'on se donne un ensemble compact $B \subset \mathbb{R}^n$ et un sous-groupe L du groupe d'homologie de Čech $\check{H}_{d-1}(B;G)$ de dimension $(d-1)$ sur un groupe abelien G, on montre qu'il existe un ensemble compact $E \supset B$ tel que L est contenu dans le noyau de l'homomorphisme $\check{H}_{d-1}(B;G) \to \check{H}_{d-1}(E;G)$ induit par l'application d'inclusion $B \to E$, et pour lequel la mesure de Hausdorff $\mathcal{H}^d(E \setminus B)$ est minimale (sous ces contraintes).

Ensuite, on montre au chapitre 4, que pour tout ensemble presque minimal glissant E de dimension 2, dans un domaine régulier Σ ressemblant localement à un demi espace, associé à la frontière glissante $\partial\Sigma$, et tel que $E \supset \partial\Sigma$, il se trouve qu'à la frontière E est localement équivalent, par un homéomorphisme biHöldérien qui préserve la frontière, à un cône minimal glissant contenu dans un demi plan Ω, avec frontière glissante $\partial\Omega$. De plus les seuls cônes minimaux possibles dans ce cas sont $\partial\Omega$ seul, ou son union avec un cône de type $\mathbb{P}_+$ ou $\mathbb{Y}_+$.

Abstract

This thesis focuses on the existence and regularity of minimal sets.

First we show, in Chapter 3, that there exists (at least) a minimizer for Reifenberg Plateau problems. That is, Given a compact set $B \subset \mathbb{R}^n$, and a subgroup L of the Čech homology group $\check{H}_{d-1}(B;G)$ of dimension $(d-1)$ over an abelian group G, we will show that there exists a compact set $E \supset B$ such that L is contained in the kernel of the homomorphism $\check{H}_{d-1}(B;G) \to \check{H}_{d-1}(E;G)$ induced by the natural inclusion map $B \to E$, and such that the Hausdorff measure $\mathcal{H}^d(E \setminus B)$ is minimal under these constraints.

Next we will show, in Chapter 4, that if E is a sliding almost minimal set of dimension 2, in a smooth domain Σ that looks locally like a half space, and with sliding boundary $\partial\Sigma$, and if in addition $E \supset \partial\Sigma$, then, near every point of the boundary $\partial\Sigma$, E is locally biHölder equivalent to a sliding minimal cone (in a half space Ω, and with sliding boundary $\partial\Omega$). In addition the only possible sliding minimal cones in this case are $\partial\Omega$ or the union of $\partial\Omega$ with a cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$.

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Chapitre 1
Introduction

L'étude des propriétés géométriques des ensembles est centrale dans la thérie géométrique de la mesure. En particulier, l'existence et la régularité d'ensembles ayant des propriétés de minimalité sont des sujets très populaires. Souvent ces ensembles sont associés à des notions physiques fortes. Par exemple, les films de savon sont bien modélisés par les ensembles minimaux au sens d'Almgren (définis ci-dessous). La théorie géométrique de la mesure est née de l'étude du problème de Plateau, qui vient cependant de l'observation expérimentale par un physicien, Joseph Plateau, qui observa que lorsque l'on plonge un fil de fer dans de l'eau savonneuse, puis qu'on le retire, la forme qu'on obtient est un ensemble minimal porté par le fil. En gros le problème de Plateau est celui de l'existence, puis de la régularité, d'ensemble minimaux avec une contrainte de frontière.

1.1 Existence

Au cours du 20ième siècle, le problème de Plateau a été résolu avec de nombreuses modélisations différentes. La première solution a été donnée par Jesse Douglas [15] and Tibor Radó [37], en décrivant les films comme des images paramétrées d'un disque. Leur manière d'énoncer le problème est célèbre ; cependant les solutions ne tiennent pas compte de ce qui se produit lorsque deux surfaces se croisent, et oublient les surfaces non orientables, de sorte que certains types de films de savon ne sont pas représentés.

La manière la plus populaire d'énoncer et de résoudre le problème de Plateau est d'utiliser les ensembles de périmètre fini (De Giorgi) et les courants (Federer and Fleming). En particulier, Federer and Fleming [20] ont introduit les courants dans le sujet. Ils ont démontré un théorème d'existence très général, pour un courant intégral S dont la masse est minimale sous la contrainte de bord $\partial S = T$ (où T est un courant intégral donné de bord nul), comme conséquence d'un résultat de compacité. Les courants minimaux (pour la masse) ont une théorie de régularité très riche ; voir [30] pour une bonne description du sujet.

Reifenberg [38] a utilisé l'homologie de Čech pour décrire le problème de

Plateau comme suit. Soient $B \subset \mathbb{R}^n$ un ensemble compact (qu'on voit comme un bord) $d \in (0, n)$ un entier, et G un groupe abélien. On se donne aussi un sous-groupe L du groupe d'homologie de Čech $\check{H}_{d-1}(B; G)$ de dimension $d-1$ sur B. On dit que l'ensemble compact $S \subset \mathbb{R}^n$ a un bord algébrique qui contient L quand $B \subset S$ et L est contenu dans le noyau de l'homomorphisme $\check{H}_{d-1}(B; G) \to \check{H}_{d-1}(S; G)$ induit par l'inclusion $i_{B,S} : B \to S$. Reifenberg [38] a démontré que quand B est un ensemble compact de dimension $(d-1)$ et G est un groupe abélien compact, alors il existe toujours au moins un ensemble compact $S \supset B$, qui a un bord algébrique contenant L, et dont la mesure de Hausdorff d-dimensionnelle est minimale (sous cette contrainte). On appelle S un minimiseur de Reifenberg homologique.

Plus récemment Thierry De Pauw [34] a démontré l'existence de minimiseurs aussi quand $G = \mathbb{Z}$ est le groupe des entiers relatifs, $n = 3$, $d = 2$, et B est formé de courbes lisses.

Nous allons généraliser les résultats de Reifenberg à tout groupe abélien G et tout ensemble compact B : nous montrerons que pour tout choix de groupe abélien G et d'ensemble compact $B \subset \mathbb{R}^n$, et pour tout sous-groupe $L \subset \check{H}_{d-1}(B; G)$, il existe un compact S, qui a un bord algébrique contenant L, et qui minimise $\mathcal{H}^d(S \setminus B)$ parmi les ensembles compacts qui ont un bord algébrique contenant L. De plus nous pourrons remplacer la mesure de Hausdorff $\mathcal{H}^d(S \setminus B)$ par $J_F(S \setminus B)$, l'intégrale sur $S \setminus B$ d'un "intégrant elliptique généralisé" F. Disons quelques mots de la notation J_F et des intégrants elliptiques ; les définitions précises se trouvent au paragraphe 3.1, et l'énoncé correspondant est le théorème 3.19. On note $G(n, d)$ la Grassmannienne des sous-espaces vectoriels de dimension d de l'espace euclidien $\mathbb{R}^n$. On pose, lorsque E est de mesure de Hausdorff $\mathcal{H}^d(E)$ finie et $F : \mathbb{R}^n \times G(n, d) \to \mathbb{R}^+$ est positive,

$$J_F(E) = \int_{x \in E_{rec}} F(x, T_x E)\, d\mathcal{H}^d(x) + \int_{x \in E_{irr}} \Big\{ \sup_{\pi \in G(n,d)} F(x, \pi) \Big\}\, d\mathcal{H}^d(x),$$

où E_{rec} et E_{irr} sont les parties rectifiable et purement non rectifiable de E. Un intégrannt elliptique généralisé est une fonction positive $F : \mathbb{R}^n \times G(n, d) \to \mathbb{R}^+$ telle que

$$0 < c_1 \leq F \leq c_1 < +\infty, \text{ pour certaines constantes } c_1, c_2 > 0,$$

et

$$J_F(\pi \cap B(x, r)) \leq J_F(S) + h(r) r^d$$

pour tout d-plan affine π passant par x et tout ensemble rectifiable compact $S \subset \overline{B(x, r)}$ qui contient $\pi \cap \partial B(x, r)$ mais ne peut pas être envoyé dans

$\pi \cap \partial B(x,r)$ par une application Lipschitzienne $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ telle que $\varphi|_{\pi\cap\partial B(x,r)} = \mathrm{id}$; ici $h : [0,+\infty] \to [0,\infty]$ est une fonction croissante donnée à l'avance telle que $\lim_{t\to 0} h(t) = 0$.

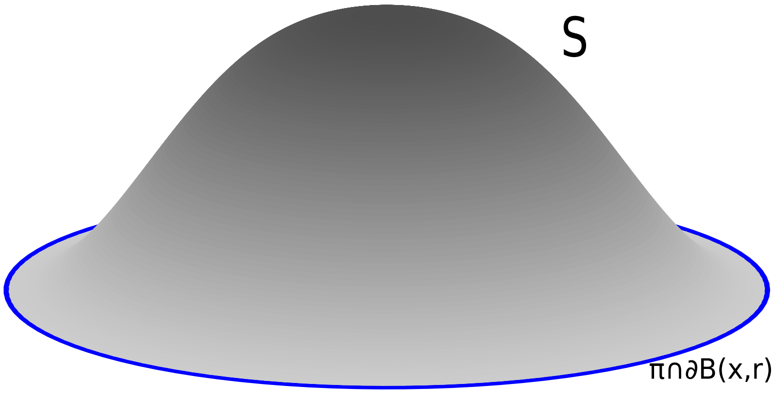


FIGURE 1.1 : $J_F(\pi \cap B(x,r)) \leq J_F(S) + h(r)r^d$

Pour l'auteur, les minimiseurs de Reifenberg homologiques donnent souvent une meilleure description des films de savon que les courants minimiseurs de masse, et ils sont plus proches des (supports fermés des) courants minimiseurs de taille, qui sont les courants S qui minimisent la quantité $Size(S)$ sous la même condition $\partial S = T$ que plus haut, mais où la taille $Size(S)$ est, en gros, la mesure de Hausdorff de l'ensemble où la fonction multiplicité qui définit S comme courant intégral est non nulle. Ainsi, la masse de S compte la multiplicité, mais pas la taille. On renvoie à [19, 20, 34] pour des définitions précises, et un compte-rendu plus détaillé du problème de Plateau pour les minimiseurs de taille. Faisons juste deux observations ici, en rapport avec le problème de Reifenberg. La Figure 1.2 représente le support d'un courant minimiseur de taille, mais pas de masse (la multiplicité sur le cercle central est 2, donc la masse est plus grande que la taille).

Même quand le courant de bord T est le courant d'intégration sur une courbe lisse (mais possiblement nouée) dans $\mathbb{R}^3$, il n'y a pas de théorème général d'existence pour les courants minimiseurs de taille. Cependant Frank Morgan [30] a démontré l'existence d'un courant minimiseur de taille lorsque le bord est une variété lisse contenue dans la frontière d'un corps convexe, et Thierry de Pauw and Robert Hardt [35] ont démontré l'existence de courants qui minimisent des énergies qui se situent entre masse et taille (typiquement, obtenues en intégrant une petite puissance de la multiplicité).

La raison pour laquelle la démonstration usuelle (par compacité) d'existence pour les courants minimiseurs de masse ne passe pas aux minimiseurs de taille est que la taille de S ne donne aucun contrôle sur la multiplicité,

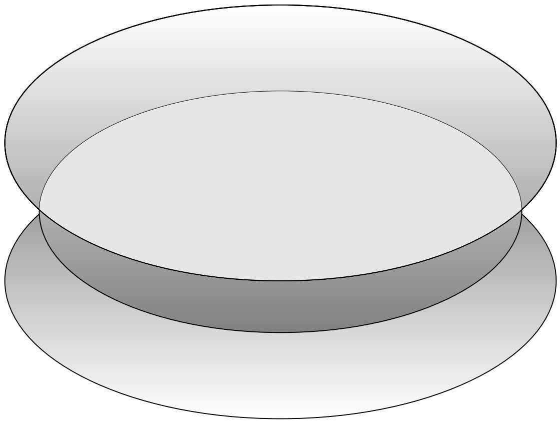


FIGURE 1.2 : courant minimiseur de taille, mais pas de masse

de sorte que la limite d'une suite minimisante pourrait très bien ne pas avoir une masse finie, donc ne même pas exister en tant que courant. Cette observation est liée à la raison pour laquelle Reifenberg s'est restreint aux groupes compacts (de sorte que les multiplicités ne tendent pas vers l'infini).

Dans [2], F. Almgren propose une méthode pour prouver le théorème de Reifenberg, et même l'étendre à des groupes G quelconques et des intégrants elliptiques assez généraux. La méthode utilise les varifolds (alors récemment découverts), ou les chaînes bémol, et un argument de feuilles multiples pour résoudre le problème des grandes multiplicités. L'argument est aussi fort subtil et elliptique. Incidemment, Almgren utilise les groupes d'homologie relative de Vietoris H_d^v au lieu des groupes d'homologie de Čech. Dans son papier, le bord B est un ensemble compact $(d-1)$-rectifiable dans $\mathbb{R}^n$, tel que $\mathcal{H}^{d-1}(B) < +\infty$, et une surface est un ensemble compact d-rectifiable $S \subset \mathbb{R}^n$. Pour tout $\sigma \in H_d^v(\mathbb{R}^n, B; G)$, on dit que S engendre σ quand $i_k(\sigma) = 0$, où l'on note $H_d^v(\mathbb{R}^n, B; G)$ le d-ième groupe d'homologie relative de Vietoris de la paire $(\mathbb{R}^n, B)$, et où

$$i_k : H_d^v(\mathbb{R}^n, B; G) \to H_d^v(\mathbb{R}^n, B \cup S; G)$$

est l'homomorphisme induit par l'inclusion $i : B \to B \cup S$. Signalons que Dowker, in [16, Theorem 2a] a démontré que les groupes d'homologie de Čech et Vietoris sur un groupe abélien G sont isomorphes pour des espaces topologiques quelconques.

Le problème de Plateau homologique est clairement lié au problème des minimiseurs de taille, et par exemple T. De Pauw [34] montre que dans le cas simple où B est une courbe lisse, les infimums pour les deux problèmes sont égaux. Dans le même papier, T. De Pauw étend également le théorème de

Reifenberg (pour des courbes dans $\mathbb{R}^3$) au groupe $G = \mathbb{Z}$. Malheureusement, bien que sa démonstration utilise de la minimisation de courants, elle ne donne pas l'existence d'un courant minimiseur de taille (il faudrait encore construire un courant sur l'ensemble minimal obtenu).

Soient $\mathscr{C}$ une collection d'ensembles compacts et F un intégrant. On renvoie au début du chapitre 3 pour des définitions précises des intégrants, des intégrants elliptiques généralisés, et de l'intégrale $J_F(E)$ d'un intégrant F sur l'ensemble E.

On pose

$$m(\mathscr{C}, F) = \inf\{J_F(E \setminus B) \mid E \in \mathscr{C}\}.$$

Au chapitre 3.2, on démontre le résultat d'existence général suivant.

Theorem 3.17. *Soient F un intégrant elliptique généralisé et $\mathscr{C}$ une classe de parties compactes de $\mathbb{R}^n$. On suppose que $\mathscr{C}$ vérifie les conditions suivantes :*

(1) *Pour toute fonction Lipschitzienne $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ telle que $\varphi|_B = \mathrm{id}_B$ et tout $E \in \mathscr{C}$, $\varphi(E) \in \mathscr{C}$;*

(2) *Pour toute suite $\{E_k\}_{k=1}^{\infty} \subset \mathscr{C}$ telle que $E_k \to E$ en distance de Hausdorff, $E \in \mathscr{C}$.*

Alors il existe $E \in \mathscr{C}$ tel que $J_F(E \setminus B) = m(\mathscr{C}, F)$.

En fait, l'existence de minimiseurs homologiques de Reifenberg est une conséquence immediate de ce théorème. En effet, soient $B \subset \mathbb{R}^n$ un ensemble compact, F intégrant elliptique généralisé, G un groupe abélien, et L un sous-groupe de $\check{H}_{d-1}(B; G)$. Notons $\mathscr{C}_{\text{Čech}}(B, G, L)$ la collection des ensembles compacts E tels que $B \subset E$ et L est contenu dans le noyau de l'homomorphisme $\check{H}_{d-1}(B; G) \to \check{H}_{d-1}(E; G)$ induit par l'application d'inclusion $B \to E$.

Theorem 3.19. *Il existe un ensemble compact $E \in \mathscr{C}_{Čech}(B, G, L)$ tel que*

$$J_F(E \setminus B) = m\left(\mathscr{C}_{Čech}(B, G, L), F\right).$$

La démonstration du théorème repose sur deux développements récents qui lui permettent de fonctionner plus facilement et d'éviter les problèmes de grande multiplicité.

Le premier développement est un lemme introduit par Dal Maso, Morel, et Solimini [28] dans le contexte de la fonctionnelle de Mumford-Shah, et qui donne une condition suffisante, portant sur une suite d'ensembles E_k qui converge en distance de Hausdorff vers une limite E, pour qu'on ait l'inégalité de semicontinuité inférieure

$$\mathcal{H}^d(E) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k). \tag{1.1.1}$$

Ce résultat sera très pratique quand on cherchera un minimiseur à l'aide d'une suite minimisante. Les hypothéses originales du lemme de [28] ne sont pas toujours agréables à vérifier, mais heureusement Guy David [7] a décrit des moyens de l'appliquer à une suite d'ensembles quasiminimaux. Il a démontré que si $\{E_k\}$ est une suite d'ensembles quasiminimaux (réduits) avec une même constante de quasiminimalité, alors la limite de Hausdorff de la suite est aussi quasiminimale, avec la même constante de quasiminimalité, et de plus l'inégalité de semicontinuité (2.1.1) est satisfaite. Voir le théorème 3.4 dans [7]. La démonstration utilise le lemme de [28].

Ici on veut non seulement trouver un ensemble qui minimise la mesure de Hausdorff, mais aussi obtenir le même résultat pour l'intégrale d'un intégrant. Pour ceci il semble crucial de disposer d'une inégalité de semicontinuité inférieure pour l'intégrale d'un intégrant. Heureusement, pour une classe assez large d'intégrants, on peut obtenir l'inégalité de semicontinuité

$$J_F(E) \leq \liminf_{k\to+\infty} J_F(E_k), \tag{1.1.2}$$

où l'on renvoie au chapitre 3.1 pour une définition précise de $J_F(E)$, et au théorème 3.8 pour un énoncé precis de l'inégalité de semicontinuité. Signalons que notre démonstration du théorème 3.8 (pour l'inégalité 2.1.2) est fort différente de la preuve du lemme de [28] ou du théorème 3.4 dans [7]. Par ailleurs, il ne faut pas espérer que l'inégalité (2.1.2) soit vraie pour tout intégrant. Pour certains intégrants, on trouve aisément une suite d'ensembles quasiminimaux telle que (2.1.2) est fausse ; voir l'exemple en fin de paragraphe 3.1.3.

Mais l'outil principal pour notre démonstration du théorème 3.17 sera un résultat récent de V. Feuvrier [22], où il construit des réseaux polyédriques adaptés à un ensemble rectifiable donné (penser aux cubes dyadiques usuels, mais où l'on se débrouille pour que les faces soient souvent presque parallèles à l'ensemble) qui lui permettent de construire une suite minimisante composée d'ensembles qui sont localement uniformément quasiminimaux, de sorte qu'on peut lui appliquer notre inégalité de semicontinuité.

Une telle construction a été utilisée par Xiangyu Liang [26], pour démontrer des résultats d'existence d'ensembles minimisant la mesure de Hausdorff sous des conditions homologiques qui généralisent des contraintes de séparation (en codimensions supérieures à 1).

Donnons une idée de la démonstration du théorème 3.17. On part d'une suite minimisante $\{E_k\} \subset \mathscr{C}$. On utilise une technique découverte par Vincent Feuvrier, qui permet de construire une nouvelle suite $\{E_k''\}$, où E_k'' est un compétiteur de E_k,

$$J_F(E_k'' \setminus B) \leq (1 + 2^{-k}) J_F(E_k \setminus B),$$

et avec des propriétés supplémentaires (quasiminimalité uniforme) qui permettent d'appliquer un résultat de semicontinuité inférieure (le théorème 3.8) et montrer que l'ensemble limite $E = \lim E''_k$ est un minimiseur. Pour le théorème 3.19, on doit juste montrer que $\mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L)$ vérifie les deux conditions du théorème 3.17. La première est claire, et la second vient de la propriété de continuité de l'homologie de Čech vis à vis des limites inverses.

1.2 Régularité

Dans [9, 10], Guy David a proposé d'étudier une variante du problème de Plateau, avec des conditions de bord glissantes. On se donne un ensemble fermé $B \subset \mathbb{R}^n$, et un ensemble initial fermé $E_0 \supset B$. Un compétiteur de E_0 dans l'ouvert U est un ensemble $\varphi_1(E_0)$, où $\{\varphi_t\}_{0 \leq t \leq 1}$ est une famille d'applications de $E \cap U$ dans U telle que $\varphi_0 = \mathrm{id}$, $\varphi_t(E \cap B) \subset B$, la fonction $(x, t) \mapsto \varphi_t(x)$ est continue sur $E \cap U \times [0, 1]$ et coincide avec l'identité hors d'une partie compacte de $E \times [0, 1]$; on renvoie au chapitre 4 pour les détails. On veut trouver un compétiteur E dont la mesure $\mathcal{H}^d(E \setminus B)$ soit minimale. La condition au bord $\varphi_t(E \cap B) \subset B$ semble très naturelle pour le problème de Plateau (décrire des films de savon). L'un de ses avantages (par rapport à fixer les points de B) est qu'il semble alors plus facile de prouver des propriétés de régularité au bord. En fait, [11] est une préparation à la démonstration de telles propriétés.

Le théorème 3.17 ne donne aucun résultat d'existence pour le problème de Plateau avec conditions glissantes au bord. Mais dans [25, 36], les auteurs proposent une approche directe au problème de Plateau. Ils prouvent un résultat d'existence, qui dit que quand B est un ensemble fermé tel que $\mathcal{H}^d(B) = 0$, il existe (au moins) un minimiseur pour le problème de Plateau avec conditions glissantes au bord (et en prenant pour E_0 l'ensemble lui-même). Une autre approche, plus désagréable, est la suivante : commencer par prouver que les ensembles minimaux glissants ont assez de régularité au bord pour être la cible de rétractions lipschitziennes dans un voisinage, puis appliquer ceci à la limite d'une suite minimisante pour montrer qu'elle est encore un compétiteur du E_0 initial.

Il est important et intéressant d'étudier la régularité des films de savon. Joseph Plateau déclara que les surfaces formées par des bulles de savon ne sont en contact que de deux manières possibles : soit trois surfaces se rencontre, avec un angle de 120° le long d'une courbe, soit six surfaces se rencontrent en un sommet, en faisant des angles de $3 \arccos(-\frac{1}{3}) \approx 109°$. C'est ce qu'a prouvé Jean Taylor [39]. Elle a démontré que tout ensemble presque minimal de dimension 2 dans $\mathbb{R}^3$ est localement C^1-équivalent à un cône minimal dans $\mathbb{R}^3$, et que les seuls cônes minimaux dans $\mathbb{R}^3$ sont les plans, et les cônes de type $\mathbb{Y}$ (voir la Figure 1.3), et de type $\mathbb{T}$ (Figure 1.4).

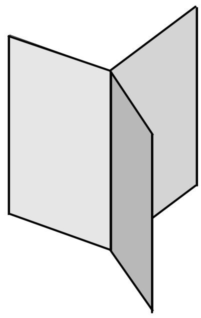


FIGURE 1.3 : un cône de type $\mathbb{Y}$

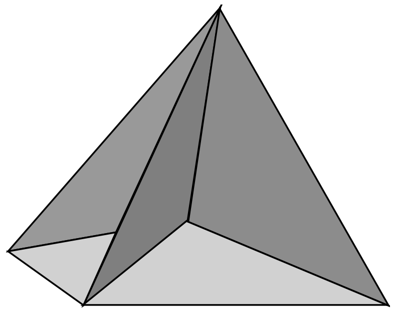


FIGURE 1.4 : un cône de type $\mathbb{T}$

Ainsi, pour certaines solutions du problème de Plateau, le théorème de Jean Taylor's donne une description claire et élégante de leur comportement loin du bord. Par contre on sait trés peu de choses sur leur comportement au bord. En fait, il y a assez peu de résultats de régularité qui vont jusqu'au bord. Dans [1], on trouve un résultat pour les varifolds ; dans [23], un résultat pour les solutions d'un problème de Plateau spécifique ; dans [41], un résultat pour les courants qui minimisent la masse. Citons également un résultat de J. Taylor [40] sur le comportement au bord de chaînes bémol minimisantes, ou de manière équivalente d'ensembles de périmètre finis minimisants. On trouve également dans [24] ou [32] une conjecture concernant les divers types de singularité d'un film de dimension 2 près de la frontière. On est encore loin de résoudre cette conjecture.

Dans [8] et [6], Guy David donne une démonstration nouvelle et plus détaillée d'une bonne partie du résultat de régularité de Jean Taylor's sur les ensembles presque minimaux au sens d'Almgren, de dimension 2 dans $\mathbb{R}^3$, et généralise le résultat à $\mathbb{R}^n$ (mais avec seulement une équivalence Höldérienne). En même temps, il prouve un théorème de presque monotonie de la densité pour les ensembles presque minimaux (loin du bord). En fait, sa démonstration de régularité Höldérienne repose sur la propriété de presque monotonie et un théorème de paramétrage de Reifenberg. Dans [11, Part VI : Monotone density], il démontre un résultat semblable (presque monotonie de la densité) mais au bord pour un ensemble presque minimal glissant. C'est ce qui nous permettra de démontrer un résultat de régularité Höldérienne jusqu'au bord, pour certains ensembles presque minimaux glissants. C'est le théorème suivant.

Theorem 4.31. *Soit $\Sigma \subset \mathbb{R}^3$ un ensemble fermé connexe dont la frontière $\partial\Sigma$ est une variété de classe C^1 de dimension 2. Soient x un point de $\partial\Sigma$,*

U un voisinage de x, et $E \subset \Sigma$ un (U,h)-ensemble presque minimal glissant associé à la frontière $\partial\Sigma$ et tel que $E \supset \partial\Sigma$. Alors pour tout petit $\tau > 0$, on peut trouver un rayon $\rho > 0$, un cône minimal glissant Z dans un demi espace Ω associé à la frontière $L_1 = \partial\Omega$, et enfin un homéomorphisme biHöldérien $\phi : B(x, 3\rho/2) \cap \Omega \to B(x, 2\rho) \cap \Sigma$ tel que

$$\phi(x) \in \partial\Sigma \text{ pour } x \in L_1 \cap B(x, 3\rho/2), \ \|\phi - \mathrm{id}\|_\infty \le 3\tau\rho,$$
$$C\,|z-y|^{1+\tau} \le |\phi(z) - \phi(y)| \le C^{-1}\,|z-y|^{\frac{1}{1+\tau}},$$
$$B(x,\rho) \cap \Sigma \subset \phi\left(B\left(x, \frac{3\rho}{2}\right) \cap \Omega\right) \subset B(x, 2\rho) \cap \Sigma,$$
$$E \cap B(x,\rho) \subset \phi\left(Z \cap B\left(x, \frac{3\rho}{2}\right)\right) \subset E \cap B(x, 2\rho).$$

La liste des cônes minimaux glissants dans le demi espace Ω associés à la frontière $L_1 = \partial\Omega$, et qui contiennent L_1, n'est pas compliquée. Ce sont le cône L_1 seul, et les cônes $L_1 \cup Z$, où Z est un cône de type $\mathbb{P}_+$ ou $\mathbb{Y}_+$. Voir le chapitre 4.2 pour des définitions précises, et le théorème 4.15 pour un énoncé.

Notre hypothèse que $E \supset \partial\Sigma$ semble une condition raisonnable pour des films de savon. Quand on trempe un fil de fer dans de l'eau savonneuse, puis qu'on le retire, on obtient (souvent) un film de savon ; le fil est notre frontière glissante et le film serait modélisé par un ensemble presque minimal glissant. En fait la surface semble bien contenir le fil (tout le fil est mouillé). Ainsi notre hypothèse que $E \supset \partial\Sigma$ semble naturelle à l'auteur.

Il serait aussi très intéressant de considérer la régularité au bord des ensembles presque minimaux glissants, sans la contrainte de contenir le bord. Mais malheureusement, sans supposer que $E \supset \partial\Sigma$, nous n'avons pas de résultat satisfaisant. C'est parce que dans ce cas, les limites par explosion de E en un point du bord peuvent aussi être des cônes de type $\mathbb{T}_+$ (voir le chapitre 4.2 pour définition, et la figure 1.5) ou de type $\mathbb{V}$ (cette fois voir [12, p. 9], et la figure 2.6).

Quand E a une limite par explosion de type $\mathbb{V}$, on s'attend à des difficultés sérieuses pour la démonstration (voir la figure 1.7, qui est un exemple potentiel de film de savon, pour lequel la régularité semble difficile à démontrer).

Pour tout $\lambda \in [0,1]$, on considère la mesure $\mu_\lambda = \mathcal{H}^2 - (1-\lambda)\mathcal{H}^2 \llcorner \partial\Sigma$ définie par $\mu_\lambda(E) = \lambda\mathcal{H}^2(E \cap \partial\Sigma) + \mathcal{H}^2(E \setminus \partial\Sigma)$ pour tout ensemble $E \subset \mathbb{R}^3$. On définit les ensembles presque minimaux pour μ_λ comme on l'a fait dans la définition 4.2, mais en remplaçant la mesure de Hausdorff usuelle par μ_λ ; on dira dans la suite que ces ensembles sont (U,h,μ_λ) presque minimaux glissants. Le théorème ci-dessus dit que tout ensemble (U,h,μ_0) presque minimal

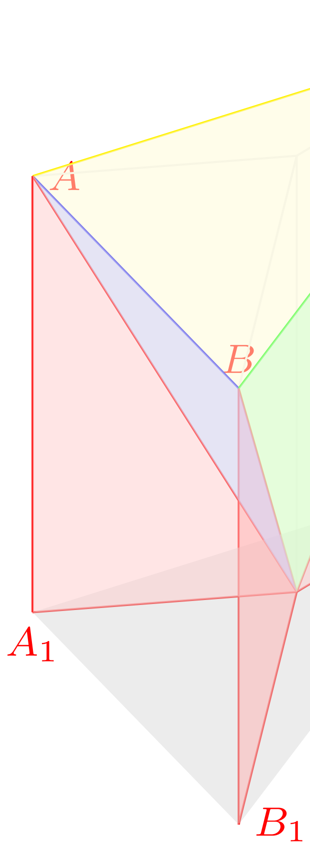


FIGURE 1.5 : cône de type $\mathbb{T}_+$

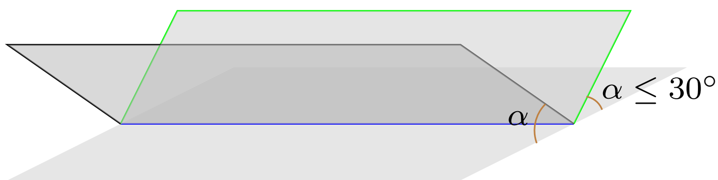


FIGURE 1.6 : cône de type $\mathbb{V}$

glissant (dans Σ, et avec la frontière glissante $\partial\Sigma$) est localement biHölder-équivalent à un cône minimal glissant (pour la mesure μ_0 et dans un demi espace Ω avec la frontière glissante $\partial\Omega$). Une question naturelle est de savoir si tout ensemble (U, h, μ_λ) presque minimal glissant est localement biHölder-équivalent à un cône minimal glissant (pour la mesure μ_λ). En fait, le cas

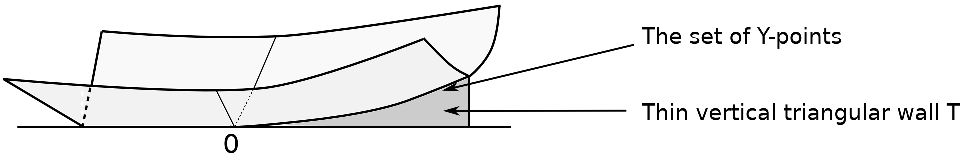


FIGURE 1.7 : Une limite par explosion en 0 est un cône de type $\mathbb{V}$

où $\lambda = 1$ est équivalent à ce dont on a parlé ci-dessus, à savoir la régularité au bord des ensembles presque minimaux glissants sans supposer qu'ils contiennent la frontière, et la question semble difficile. Quand $0 < \lambda < 1$, il faut d'abord donner la liste des cônes minimaux pour la mesure μ_λ, et nous ne souhaitons pas élaborer ici.

Jean Taylor [40, Theorem 5] a démontré des résultats de cet ordre (et en fait plus précis), en étudiant la régularité au bord pour des problèmes de capillarité. La situation était quand même différente, parce qu'elle considérait des chaînes bémol (ou de manière équivalente, d'ensembles de Caccioppoli) ; La situation était quand même différente ; en particulier, $\mathbb{Y}_+$ (qui sépare l'espace en trois composantes) n'intervient pas comme cône tangent dans son étude.

Chapter 2
Introduction

English version

The study of geometric properties of sets is the core topic in geometric measure theory. In particular, the study of the existence and regularity of sets with some minimal properties is very popular. Usually, these sets enjoy a strong physics background. For example, soap films are well modeled by Almgren minimal sets (defined below). The geometric measure theory is born out of solving Plateau's problem. However Plateau's problem was introduced by the physicist Joseph Plateau through experimental observation. Plateau noticed that if we dip a wire into some soapy water, when we pull it out we obtain a minimal set created by the soap film. Roughly speaking, Plateau's problem is to show the existence and then the regularity of minimal sets under some boundary constraints.

2.1 Existence

During the 20^{th} century Plateau's problem was solved with many different modelings. The first solution was given by Jesse Douglas [15] and Tibor Radó [37]. This is a celebrated way to state the Plateau problem, where surfaces are considered as parametrized images of a disk, but the solutions do not account for crossing surfaces and nonorientable surfaces, thus some reasonable forms of soap film are not taken into account.

The most popular way to state and prove existence results for Plateau's problem has been through sets of finite perimeter (De Giorgi) and currents (Federer and Fleming). In particular, Federer and Fleming [20] introduced currents in the subject. A very general existence result for an integral current S whose mass is minimal under the boundary constraint $\partial S = T$ immediately follows from a compactness property, where T is a given integral current such that $\partial T = 0$. Mass-minimizing currents have a very rich regularity theory; we refer to [30] for a nice overview.

Reifenberg [38] used Čech homology to depict Plateau's problem. Let a compact set $B \subset \mathbb{R}^n$ (considered as boundary) and an integer $d \in (0, n)$ be given. Also let G be an abelian group, and pick a subgroup L of the $(d-1)$ dimensional Čech homology group $\check{H}_{d-1}(B;G)$. Let $S \subset \mathbb{R}^n$; we say that S is of algebraic boundary containing L, if $B \subset S$ and L is contained in the kernel of the homomorphism $\check{H}_{d-1}(B;G) \to \check{H}_{d-1}(S;G)$ induced by the inclusion map $i_{B,S} : B \to S$. Reifenberg [38] proved that when B is a $(d-1)$ dimensional compact set and G is a compact abelian group, then among all the compact sets S of algebraic boundary containing L, there exists (at least) one with smallest d-dimensional Hausdorff. We call such a set S a (Reifenberg homological) minimizer.

More recently, Thierry De Pauw [34] proved the existence of minimizers also when $G = \mathbb{Z}$ is the group of integers, $n = 3$, $d = 2$, and B is a nice curve.

We will generalize Reifenberg's results to any abelian group G and any compact set B. That is, for any abelian group G, any compact set $B \subset \mathbb{R}^n$, and any subgroup $L \subset \check{H}_{d-1}(B;G)$, we will show that there exists a compact set S which is of boundary containing L and that minimizes the quantity $\mathcal{H}^d(S \setminus B)$ among all those compact sets. Moreover, we can replace the Hausdorff measure $\mathcal{H}^d(S \setminus B)$ by $J_F(S \setminus B)$, the integral of an generalized elliptic integrand F on $S \setminus B$. We quickly introduce the notation J_F and generalized elliptic integrand, but we refer to section 3.1 for precise definitions, and to Theorem 3.19 for a precise statement. We denote by $G(n,d)$ the Grassmannian which consist of d-dimensional subspaces of euclidean space $\mathbb{R}^n$. Given a set E of finite Hausdorff measure $\mathcal{H}^d(E)$, and a positive function $F : \mathbb{R}^n \times G(n,d) \to \mathbb{R}^+$, we put

$$J_F(E) = \int_{x \in E_{rec}} F(x, T_x E)\, d\mathcal{H}^d(x) + \int_{x \in E_{irr}} \Big\{ \sup_{\pi \in G(n,d)} F(x,\pi) \Big\}\, d\mathcal{H}^d(x),$$

where E_{rec} and E_{irr} are the rectifiable and purely unrectifiable parts of E. A generalized elliptic integrand F is a positive continuous function $\mathbb{R}^n \times G(n,d) \to \mathbb{R}^+$ such that

$$0 < c_1 \leq F \leq c_1 < +\infty, \text{ for some } c_1, c_2 > 0,$$

and that

$$J_F(\pi \cap B(x,r)) \leq J_F(S) + h(r) r^d$$

for any d-plane π through x and any rectifiable compact set $S \subset \overline{B(x,r)}$ which contains $\pi \cap \partial B(x,r)$ and can not be mapped into $\pi \cap \partial B(x,r)$ by any Lipschitz map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi|_{\pi \cap \partial B(x,r)} = \mathrm{id}$, where $h : [0, +\infty] \to [0, \infty]$ is a nondecreasing function with $\lim_{t \to 0} h(t) = 0$.

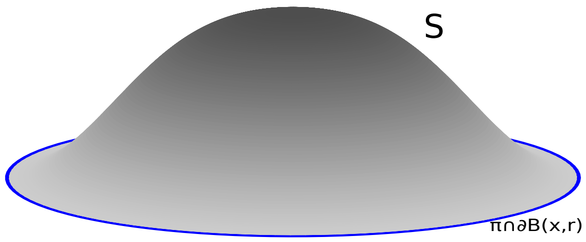


Figure 2.1: $J_F(\pi \cap B(x,r)) \leq J_F(S) + h(r)r^d$

In the author's view, Reifenberg's homological minimizers often give a better description of soap films than mass minimizers, and they are much closer to (the closed supports of) size minimizing currents. Those are currents S that minimize the quantity $Size(S)$ under a boundary constraint $\partial S = T$ as before, but where $Size(S)$ is, roughly speaking, the $\mathcal{H}^d$-measure of the set where the multiplicity function that defines S as an integral current is nonzero. Thus the mass counts the multiplicity, but not the size. We refer to [19, 20, 34] for precise definitions, and a more detailed account of the Plateau problem for size minimizing currents. We shall just mention two things here, in connection with the Reifenberg problem. Figure 2.2 depicts the support of a current which is size minimizing, but not mass minimizing (the multiplicity on the central disk is 2, so the mass is larger than the size).

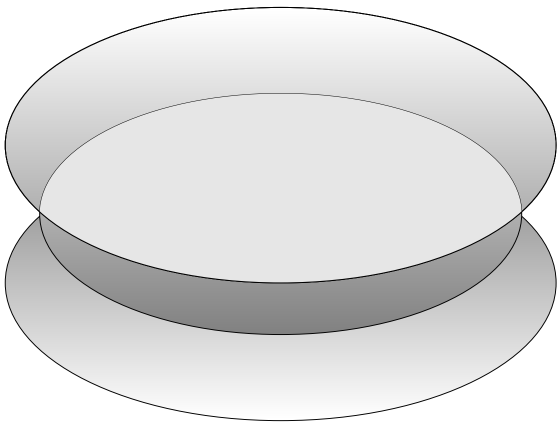


Figure 2.2: size minimizing but not mass minimizing

Even when the boundary current T is the current of integration on a smooth (but possibly linked) curve in $\mathbb{R}^3$, there is no general existence for a size minimizing current. However, Frank Morgan proved the existence of a size minimizing current [30] when the boundary is a smooth submanifold contained in the boundary of a convex body, and in [35] Thierry de Pauw and Robert Hardt proved the existence of currents which minimize energies that lie somewhere between mass and size (typically, obtained by integration of some small power of the multiplicity).

The reason why the usual proof of existence for mass minimizers, using a compactness theorem, does not work for size minimizers, is that the size of S does not give any control on the multiplicity, and so the limit of a minimizing sequence may well not have finite mass (or even not exist as a current). This issue is related to the reason why Reifenberg restricted to compact groups (so that multiplicities don't go to infinity).

In [2], F. Almgren proposed a scheme for proving Reifenberg's theorem, and even extending it to general groups and elliptic integrands. The scheme uses the then recently discovered varifolds, or flat chains, and a multiple layers argument to get rid of high multiplicities, but it is also very subtle and elliptic. Incidentally, Almgren uses Vietoris relative homology groups H_d^v instead of Čech homology groups. In his paper, a boundary B is a compact $(d-1)$-rectifiable subset of $\mathbb{R}^n$ with $\mathcal{H}^{d-1}(B) < +\infty$, and a surface S is a compact d-rectifiable subset of $\mathbb{R}^n$. For any $\sigma \in H_d^v(\mathbb{R}^n, B; G)$, a surface S spans σ if $i_k(\sigma) = 0$, where we denote by $H_d^v(\mathbb{R}^n, B; G)$ the d-th Vietoris relative homology groups of $(\mathbb{R}^n, B)$, and

$$i_k : H_d^v(\mathbb{R}^n, B; G) \to H_d^v(\mathbb{R}^n, B \cup S; G)$$

is the homomorphism induced by the inclusion map $i : B \to B \cup S$. We should mention that Dowker, in [16, Theorem 2a], proved that Čech and Vietoris homology groups over an abelian group G are isomorphic for arbitrary topological spaces.

There is some definite relation between Reifenberg's homological Plateau problem and the size minimizing currents, and for instance T. De Pauw [34] shows that in the simple case when B is a nice curve, the infimums for the two problems are equal. In the same paper, T. De Pauw also extends Reifenberg's result (for curves in $\mathbb{R}^3$) to the group $G = \mathbb{Z}$. Unfortunately, even though the proof uses minimizations among currents, this does not yet give a size minimizer (one would need to construct an appropriate current on the minimizing set).

Let $\mathscr{C}$ be a collection of compact sets and let F be an integrand. We refer to the beginning of the chapter 3 for the precise definitions of integrands, generalized elliptic integrands and the integral $J_F(E)$ of an integrand F on a set E.

We set

$$m(\mathscr{C}, F) = \inf\{J_F(E \setminus B) \mid E \in \mathscr{C}\}.$$

In Section 3.2, we will prove the following general existence result.

Theorem 3.17. *Let F be a generalized elliptic integrand, $\mathscr{C}$ a class of compact subsets in $\mathbb{R}^n$. If $\mathscr{C}$ satisfies the following conditions:*

(1) *For any Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi|_B = \mathrm{id}_B$ and any $E \in \mathscr{C}$, $\varphi(E) \in \mathscr{C}$;*

(2) *For any sequence $\{E_k\}_{k=1}^{\infty} \subset \mathscr{C}$ with $E_k \to E$ in Hausdorff distance, then $E \in \mathscr{C}$.*

Then we can find $E \in \mathscr{C}$ such that $J_F(E \setminus B) = m(\mathscr{C}, F)$.

In fact, the existence of Reifenberg homological minimizers immediately follows from the above theorem. Indeed, let the compact set $B \subset \mathbb{R}^n$, a generalized elliptic integrand F, an abelian group G, and a subgroup L of $\check{H}_{d-1}(B; G)$ be given. We denote by $\mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L)$ the collection of compact sets E which satisfy that $B \subset E$ and that L is contained in the kernel of the homomorphism $\check{H}_{d-1}(B; G) \to \check{H}_{d-1}(E; G)$ induced by the inclusion map $B \to E$.

Theorem 3.19. *There exists a compact set $E \in \mathscr{C}_{\check{C}ech}(B, G, L)$ such that*

$$J_F(E \setminus B) = m\left(\mathscr{C}_{\check{C}ech}(B, G, L), F\right).$$

The proof of the theorem relies on two recent developments that make it work more smoothly and ignore multiplicity issues.

The first development is a lemma introduced by Dal Maso, Morel, and Solimini [28] in the context of the Mumford-Shah functional, and which gives a sufficient condition, on a sequence of sets E_k that converges to a limit E in Hausdorff distance, for the lower semicontinuity inequality

$$\mathcal{H}^d(E) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k). \tag{2.1.1}$$

It will be very convenient when we seek a minimizer through a minimizing sequence. But it is not so pleasant to verify the conditions for the original lemma in [28]. Fortunately, Guy David [7] found ways to apply it to a sequence of quasiminimal sets. He proved that when $\{E_k\}$ is a sequence of quasiminimal sets with a same quasiminimal constant, then the Hausdorff limit of the sequence is also quasiminimal even with same quasiminimal constant, and the lower semicontinuity inequality (2.1.1) holds. See Theorem 3.4 in [7]. The proof uses the lemma in [28].

Here we not only want to find a minimizer for the Hausdorff measure, but also to get the same result for the integral of an integrand. For this a lower semicontinuity inequality for integral of an integrand seems crucial. Fortunately, for some kind of special integrands, the following lower semicontinuity inequality also holds :

$$J_F(E) \le \liminf_{k \to +\infty} J_F(E_k), \tag{2.1.2}$$

where we prefer to refer to Section 3.1 for a precise definition of $J_F(E)$, and to Theorem 3.8 for a precise statement of the lower semicontinuity inequality. Let us mention that our proof of Theorem 3.8 (for inequality 2.1.2) is quite different from the proof of the lemma in [28] or Theorem 3.4 in [7]. Also, don't expect that the inequality (2.1.2) holds for any integrand. For some integrands, we can easily find a sequence of quasiminimal sets such that (2.1.2) is false; see the example in the end of subsection 3.1.3.

But our main tool for the proof of Theorem 3.17 will be a recent result of V. Feuvrier [22], where he uses a construction of polyhedral networks adapted to a given set (think about the usual dyadic grids, but where you manage to have faces that are very often parallel to the given set) to construct a minimizing sequence for our problem, but which has the extra feature that it is composed of locally uniformly quasiminimal sets, to which we can apply our lower semicontinuity inequality.

Such a construction was used by Xiangyu Liang [26], to prove existence results for sets that minimize the Hausdorff measure under some homological generalization of a separation constraint (in codimension larger than 1).

Let us give a rapid sketch of the proof of Theorem 3.17. First we take a bounded minimizing sequence $\{E_k\} \subset \mathscr{C}$. Next, we use a technique found by Feuvrier to construct a new sequence $\{E_k''\}$, such that each E_k'' is a competitor of E_k,

$$J_F(E_k'' \setminus B) \le (1 + 2^{-k}) J_F(E_k \setminus B),$$

and with some extra properties (of uniform quasiminimality) that allow us to use a lower semicontinuity result (Theorem 3.8) and show that the limit set $E = \lim E_k''$ is a minimizer. For Theorem 3.19, we only need to check that $\mathscr{C}_{\check{\text{C}}\text{ech}}(B, G, L)$ satisfies the two conditions in Theorem 3.17. The first one is quit clear, and the second one comes from the continuity of Čech homology with respect to inverse limits.

2.2 Regularity

In [9, 10], Guy David proposed to consider a variant of Plateau's problem, with sliding boundary conditions. We are given a closed set $B \subset \mathbb{R}^n$, and an initial closed set $E_0 \supset B$. A competitor of E_0 is a set $\varphi_1(E_0)$, where $\{\varphi_t\}_{0 \le t \le 1}$ is a family of maps$E \cap U \to U$ such that $\varphi_0 = \mathrm{id}$, $\varphi_t(E \cap B) \subset B$, the function $(x, t) \mapsto \varphi_t(x)$ is continuous on $E \cap U \times [0, 1]$ and coincides with the identity outside of a compact set of $E \times [0,1]$; we refer to Chapter 4 for more detail. We aim to find a competitor E for which $\mathcal{H}^d(E \setminus B)$ is minimal among competitors of E_0. The sliding condition $\varphi_t(E \cap B) \subset B$ seems very natural for Plateau's problem (to describe soap films). One of its advantages (compared to fixing points of B) is that it may be easier to prove some regularity at the boundary. In fact, Paper [11] paves the way to showing the regularity.

Theorem 3.17 does not imply any existence result for the Plateau problem with sliding boundary conditions. But in [25, 36], the authors proposed a direct approach to the Plateau problem. Eventually they proved an existence result, which says that when B is a closed set with $\mathcal{H}^d(B) = 0$, then there exists (at least) a sliding minimizer for the Plateau problem with sliding boundary conditions. Another approach to the existence may be a little unpleasant: that is, first prove enough regularity for sliding minimal sets to find a Lipschitz neighborhood retraction, and then the limit set of a nice minimizing sequence will be a minimizer.

It is important and interesting to study the regularity of soap films. Joseph Plateau claimed that soap bubble surfaces always make contact in one of two ways: either three surfaces meet with 120° angles along a curve, or six surfaces meet at a vertex, forming angles of $3\arccos(-\frac{1}{3}) \approx 109°$. This was proved by Jean Taylor [39]. She proved that any two-dimensional almost minimal set in $\mathbb{R}^3$ is locally C^1-equivalent to a two-dimensional minimal cone in $\mathbb{R}^3$, and that two-dimensional minimal cones in $\mathbb{R}^3$ are planes, cones of type $\mathbb{Y}$ (see Figure 2.3), and cones of type $\mathbb{T}$ (see Figure 2.4).

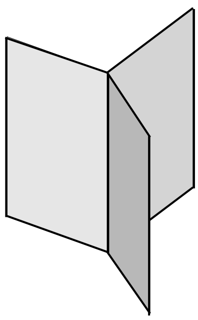


Figure 2.3: a cone of type $\mathbb{Y}$

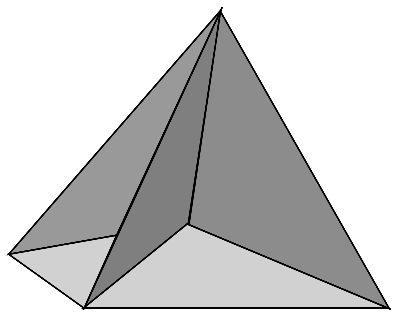


Figure 2.4: a cone of type $\mathbb{T}$

Thus, for some solutions of Plateau's problem, Jean Taylor's theorem gives a very clear and nice description of its behavior away from the boundary. But we know very little about its boundary behavior. In fact, not so many regularity results exist that go to the boundary. In [1], there is a result for varifold; in [23], there is a result for solutions of a special Plateau problem;

in [41], there is a result for mass minimizers; let us also mention a result of J.Taylor [40] that concerns the behaviour at the boundary of flat chains modulo 2, or equivalently sets of finite perimeter. In [24] or [32], there is a conjecture about the types of singularities of a soap film near the boundary. We still have a long way to achieve solving the conjecture.

In [8] and [6], Guy David gave a new, more detailed, proof of a good part of Jean Taylor's regularity theorem for Almgren almost minimal sets of dimensional 2 in $\mathbb{R}^3$, and generalized it to $\mathbb{R}^n$ (but with only a Hölder equivalence in general). At the same time, he established a theorem of almost monotonicity of density for almost minimal set away from boundary. In fact, his proof of Hölder regularity relies on the property of almost monotonicity of density and a Reifenberg parameterization. In [11, Part VI: Monotone density], he proved a similar result (a theorem of almost monotonicity of density) at the boundary for sliding almost minimal sets. This will allow us to prove the Hölder regularity of these sets at the boundary in some case. That is, the following theorem.

Theorem 4.31. *Let $\Sigma \subset \mathbb{R}^3$ be a connected closed set such that the boundary $\partial\Sigma$ is a two-dimensional C^1 submanifold. Suppose that x is a point in $\partial\Sigma$, U is a neighborhood of x, $E \subset \Sigma$ is an (U, h)-sliding-almost-minimal set with sliding boundary $\partial\Sigma$ and $E \supset \partial\Sigma$. Then for each small $\tau > 0$, we can find a radius $\rho > 0$, a sliding minimal cone Z in Ω with sliding boundary L_1 and a biHölder map $\phi : B(x, 3\rho/2) \cap \Omega \to B(x, 2\rho) \cap \Sigma$ such that*

$$\phi(x) \in \partial\Sigma \text{ for } x \in L_1 \cap B(x, 3\rho/2),\ \|\phi - \mathrm{id}\|_\infty \leq 3\tau\rho,$$
$$C\,|z-y|^{1+\tau} \leq |\phi(z) - \phi(y)| \leq C^{-1}\,|z-y|^{\frac{1}{1+\tau}},$$
$$B(x,\rho) \cap \Sigma \subset \phi\left(B\left(x, \frac{3\rho}{2}\right) \cap \Omega\right) \subset B(x, 2\rho) \cap \Sigma,$$
$$E \cap B(x,\rho) \subset \phi\left(Z \cap B\left(x, \frac{3\rho}{2}\right)\right) \subset E \cap B(x, 2\rho),$$

where $\Omega \subset \mathbb{R}^3$ is a closed half space, and L_1 is the boundary of Ω.

The list of sliding minimal cones in the half space Ω with the sliding boundary $L_1 = \partial\Omega$ and that contain L_1 is not complicated. It consist of the cones L_1 and the cones $L_1 \cup Z$, where Z is a cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$. See Section 4.2 for precise definitions and Theorem 4.15 for a precise statement.

It seems to be a reasonable condition for soap film that $E \supset \partial\Sigma$. In soap film experiments, if we dip a shape of wire into some soapy water, when we pull it out we obtain a surface created by the soap film. The wire is considered as the sliding boundary, and the surface is considered as a sliding almost minimal set. Actually, this surface seems to contain the wire. Thus the assumption $E \supset \partial\Sigma$ seems natural to the author.

It would be also very interesting to consider the regularity at the boundary of sliding almost minimal sets which we do not necessarily contain the boundary. But unfortunately, without the assumption $E \supset \partial\Sigma$, we do not have a satisfactory result. Because in this case, the blow-up limits of E at a point $x \in E \cap \partial\Sigma$ could be cones of type $\mathbb{T}_+$ (see Section 4.2 for precise definition, and see following picture 2.5) or cones of type $\mathbb{V}$ (see [12, p. 9] for precise definition, see **??** for a picture). When a blow-up limit is a cone of

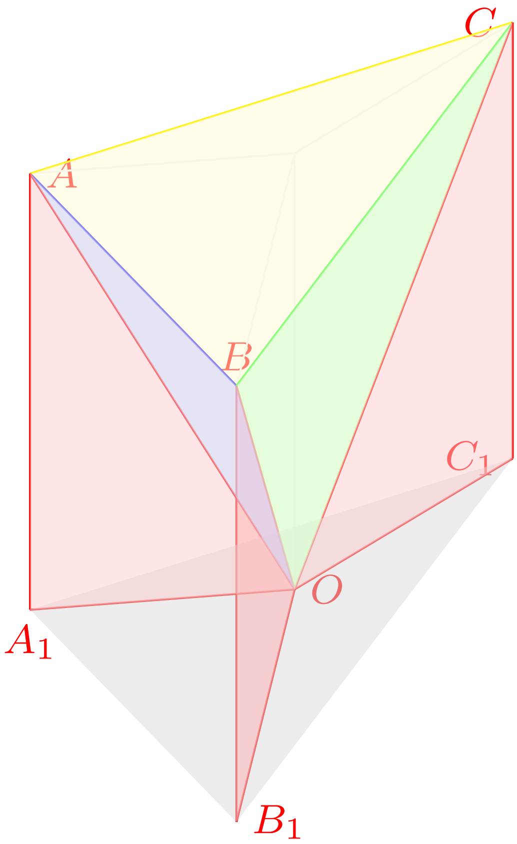


Figure 2.5: minimal cone of type $\mathbb{T}_+$

type $\mathbb{V}$, we will meet trouble (Figure 2.7 is an example of potential soap film for which regularity seems difficult to prove).

For any $\lambda \in [0,1]$, we consider the measure $\mu_\lambda = \mathcal{H}^2 - (1-\lambda)\mathcal{H}^2 \llcorner \partial\Sigma$ defined by $\mu_\lambda(E) = \lambda\mathcal{H}^2(E \cap \partial\Sigma) + \mathcal{H}^2(E \setminus \partial\Sigma)$ for any set $E \subset \mathbb{R}^3$. As in Definition 4.2, we can define sliding almost minimal sets with respect to μ_λ by replacing the usual Hausdorff measure with μ_λ; we shall call them (U, h, μ_λ)

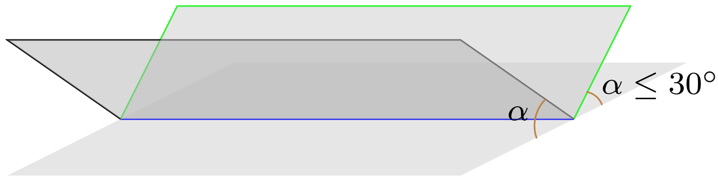


Figure 2.6: cône de type $\mathbb{V}$

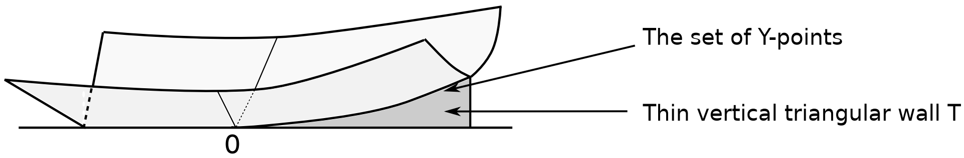


Figure 2.7: blow-up limit at 0 is a cone of type $\mathbb{V}$

sliding almost minimal. Then the above theorem is equivalent to saying that any (U, h, μ_0) sliding almost minimal set (in Σ, with sliding boundary $\partial\Sigma$) is locally biHölder equivalent to a sliding minimal cone (minimal for the measure μ_0, in a half space Ω, and with sliding boundary $\partial\Omega$). A natural question is whether any (U, h, μ_λ) sliding almost minimal set is locally biHölder equivalent to a sliding minimal (for the measure μ_λ) cone. In fact, the case when $\lambda = 1$ is equivalent to what we mentioned before, that is, the regularity at the boundary of sliding almost minimal sets which we do not necessarily contain the boundary, and it seems difficult to answer. When $0 < \lambda < 1$, the list of sliding minimal cones for measure μ_λ is not ready, so omit discussing these cases here.

Jean Taylor [40, Theorem 5] proved a very similar (and even more precise) result, that arose from getting boundary regularity for capillarity problem. The setting was different, though, because she studied the boundary of minimizing flat chains modulo 2 (or equivalently Caccioppoli sets); in particular, $\mathbb{Y}_+$, which bounds three components, does not arise as a tangent cone in her study.

Chapter 3

Existence of minimal sets under some boundary constraints

Let $n \geq 2$ be an integer, $\mathbb{R}^n$ be the n-dimensional Euclidean space with usual Euclidean structure. Let $d \in (0, n)$ be an integer. We shall denote by $\mathcal{H}^d(E)$ the d-dimensional Hausdorff measure of the Borel set $E \subset \mathbb{R}^n$. That is,

$$\mathcal{H}^d(E) = \lim_{\delta \to 0+} \mathcal{H}^d_\delta(E),$$

where

$$\mathcal{H}^d_\delta(E) = \inf \left\{ \sum_j \operatorname{diam}(U_j)^d \;\middle|\; E \subset \bigcup_j U_j,\ \operatorname{diam}(U_j) < \delta \right\},$$

i.e., the infimum is over all the coverings of E by a countable collection of sets U_j with diameters less than δ. We refer to [19, 29] for the basic properties of $\mathcal{H}^d$; notice incidentally that we could also have used the spherical Hausdorff measure, or even some more exotic variants, essentially because the competition will rather fast be restricted to rectifiable sets, for which the two measures are equal.

For any set E, any point $x \in E$ and any radius $r > 0$, we set

$$\theta_E(x, r) = \frac{\mathcal{H}^d(E \cap B(x, r))}{\omega_d r^d},$$

where ω_d denote the Hausdorff measure of d-dimensional unity ball. If the limit

$$\lim_{r \to 0} \theta_E(x, r)$$

exists, we will denote it by $\theta_E(x)$, and it will be called the density of E at the point x. When E is given, and there is no danger of confusion, we may drop the subscript E and denote it by $\theta(x)$.

A d-plane in $\mathbb{R}^n$ is a translation of a subspace of dimension d. A hyperplane in $\mathbb{R}^n$ is a $(n-1)$-plane. A set in $\mathbb{R}^n$ is called a (closed) half space if it can be written as

$$\{x + ru \mid x \in H,\ r \geq 0\},$$

where H is a hyperplane and $u \in \mathbb{R}^n$ is vector. A polyhedron of dimension n in $\mathbb{R}^n$ is a compact with non-empty interior intersection of finitely many half space. A polyhedron of dimension $k < n$ in $\mathbb{R}^n$ is a polyhedron of dimension k in a k-plane.

A set $E \subset \mathbb{R}^n$ is called d-rectifiable, if there is a sequence of Lipschitz maps $f_i : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$\mathcal{H}^d\left(E \setminus f_i(\mathbb{R}^d)\right) = 0.$$

A set E is called purely d-unrectifiable (or d-irregular) if $\mathcal{H}^d(E \cap F) = 0$ for any d-rectifiable set F. See [29, Definition 15.3] or [19, 3.2.14]. When d is clear, we may say rectifiable and purely unrectifiable instead of d-rectifiable and purely d-unrectifiable respectively.

Let $E \subset \mathbb{R}^n$ be a d-rectifiable set, $x \in E$ be any point. A d-plane π is called an approximate tangent plane if

$$\limsup_{r \to} \frac{\mathcal{H}^d(E \cap B(x, \rho))}{r^d} > 0$$

and for any $\varepsilon > 0$,

$$\lim_{r \to 0} \frac{\mathcal{H}^d(E \cap B(x, \rho) \setminus \mathcal{C}(x, \pi, r, \varepsilon))}{r^d} = 0,$$

where $\mathcal{C}(x, \pi, r, \varepsilon) = \{y \in B(x, r) \mid \operatorname{dist}(y, \pi) \leq \varepsilon |y - x|\}$.

A d-plane π is called a (true) tangent plane if for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$E \cap B(x, r) \subset \mathcal{C}(x, \pi, r, \varepsilon), \text{ for } 0 < r < r_\varepsilon.$$

We will denote by $T_x E$ the tangent plane of E at x, if it exists. Some times we may identify the tangent plane $T_x E$ with the tangent space $T_x E - x$. In particular, when we consider $T_x E$ as an element in the Grassmannian manifold $G(n, d)$, we always mean the vector space $T_x E - x$.

If $\mathcal{H}^d(E) < \infty$, then E has a decomposition into d-rectifiable and purely d-unrectifiable subsets E_{rec} and E_{irr}:

$$E = E_{rec} \cup E_{irr}.$$

See [29, Theorem 15.6]. If we suppose, in addition, that E is $\mathcal{H}^d$ measurable, then for almost every $x \in E$, the density $\theta_E(x)$ of E at x is 1, and there is a unique approximate tangent plane. We refer to [29, Chapter 15] for the more properties of rectifiable sets.

3.1 Integrands

Let $0 < d < n$ be two integers. Let $G(n, d)$ be the Grassmannian of d-dimensional subspaces of euclidean space $\mathbb{R}^n$. That is, collections of unoriented d-dimensional subspaces in $\mathbb{R}^n$ equipped with the distance d_G defined by

$$d_G(V, W) = \sup_{v \in V, |v|=1} \mathrm{dist}(v, W)$$

for any d-dimensional subspaces V and W.

An integrand is a continuous function $F : \mathbb{R}^n \times G(n, d) \to \mathbb{R}^+$ which is bounded, i.e. there exist $0 < c \le C < +\infty$ such that $c \le F(x, \pi) \le C$ for all $x \in \mathbb{R}^n$ and $\pi \in G(n, d)$.

For any d-dimensional set $E \subset \mathbb{R}^n$ with $\mathcal{H}^d(E) < \infty$, it can be written as the union of its rectifiable part and unrectifiable part, i.e. $E = E_{rec} \cup E_{irr}$, where E_{rec} is rectifiable, and E_{irr} is purely unrectifiable. For any integrand F and any positive function $f : \mathbb{R}^n \to \mathbb{R}^+$, we set

$$J_{F,f}(E) = \int_{x \in E_{rec}} F(x, T_x E_{rec}) d\mathcal{H}^d(x) + \int_{x \in E_{irr}} f(x) d\mathcal{H}^d(x).$$

For short, we denote $J_F(E) = J_{F,\tilde{F}}(E)$, where $\tilde{F}$ is a function $\mathbb{R}^n \to \mathbb{R}^+$ defined by

$$\tilde{F}(x) = \sup_{\pi \in G(n,d)} F(x, \pi).$$

A gauge function h is a nondecreasing function $[0, +\infty] \to [0, +\infty]$ such that

$$\lim_{t \to 0} h(t) = 0$$

An integrand is called generalized elliptic integrand if there exists a gauge function h such that

$$J_F(\pi \cap B(x, r)) \le J_F(S) + h(r) r^d, \tag{3.1.1}$$

for some gauge function h, where $S \subset \overline{B(x, r)}$ is any rectifiable compact set which contains $\pi \cap \partial B(x, r)$ and can not be mapped into $\pi \cap \partial B(x, r)$ by any Lipschitz $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ which leaves $\pi \cap \partial B(x, r)$ fixed.

A slightly less general class of integrands was introduced by F. Almgren [3]. That is, called elliptic integrands, integrands F which satisfies following: for each compact set $K \subset \mathbb{R}^n$, there exists a number $0 < \Gamma < \infty$ such that for each $x \in K$,

$$J_{F^x}(S) - J_{F^x}(D) \ge \Gamma[\mathcal{H}^d(S) - \mathcal{H}^d(D)],$$

whenever $D = B(0,1) \cap \pi$ for some $\pi \in B(n,d)$ and S is a compact rectifiable subset of $\mathbb{R}^n$ which cannot be mapped into $P \cap \partial B(0,1)$ by any Lipschitz mapping $\mathbb{R}^n \to \mathbb{R}^n$ which leaves $P \cap \partial B(0,1)$ fixed, where $F^x : \mathbb{R}^n \times G(n,d) \to \mathbb{R}^+$ is defined by $F^x(y,\pi) = F(x,\pi)$.

An integrand is called almost polyhedral convex, if there exists a gauge function h such that for any polyhedron Δ of dimension $(d+1)$,

$$J_F(E_1) \le \sum_{i=2}^{m} J_F(E_i) + h(r)r^d, \tag{3.1.2}$$

where $E_1, \ldots, E_m$ are (all of) the d-dimensional faces of Δ, and $r = \operatorname{diam}(\Delta)$.

Lemma 3.1. *Any generalized elliptic integrand is almost polyhedral convex with same gauge function.*

We denote by $\mathscr{L}(x,r)$ the collection of all of Lipschitz $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ which leaves $\pi \cap \partial B(x,r)$ fixed, and by $\mathscr{S}(x,r,\pi)$ the collection of all of compact set S which contains $\pi \cap \partial B(x,r)$ and can not be mapped into $\pi \cap \partial B(x,r)$ by any $\varphi \in \mathscr{L}(x,r)$.

Proof. If an integrand F satisfies (3.1.1) for some gauge function h, and any $S \in \mathscr{S}(x,r,\pi)$, we will show that F is a generalized elliptic integrand. For any $(d+1)$-dimensional polyhedron Δ, we denote by $E_1, E_2, \cdots, E_m$ the faces of Δ. We take $x \in E_1$, and let π be a d-plane through x such that $E_1 \subset \pi$. Then $\Delta \subset B(x,r_0)$, where $r_0 = \operatorname{diam}(\Delta)$. We can see that

$$S = E_2 \cup \cdots \cup E_m \cup \left(\overline{B(x,r_0)} \cap \pi \setminus E_1\right) \in \mathscr{S}(x,r_0,\pi).$$

Thus

$$J_F(\overline{B(x,r_0)} \cap \pi) \le J_F(S) + h(r_0)r_0^d.$$

Hence

$$J_F(E_1) \le J_F(E_2 \cup \cdots \cup E_m) + h(r_0)r_0^d \le \sum_{i=2}^{m} J_F(E_i) + h(r_0)r_0^d.$$

□

Lemma 3.2. *If $d = 1$, then any almost polyhedral convex integrand $F : \mathbb{R}^n \times G(n,d) \to \mathbb{R}$ is a generalized elliptic integrand.*

We will give a proof of this lemma at the end of subsection 3.1.2. We have a chance to show that polyhedral convex integrands are also generalized elliptic when $d = n-1$, but the proof will be long and boring, so we omit it here.

3.1.1 Lower semicontinuity for integral of an integrand on quasiminimal sequence

In this subsection, we will prove a lower semicontinuity inequality as we mentioned before. Let's begin with some definitions.

Definition 3.3. A polyhedral complex $\mathcal{S}$ is a finite set of closed convex polyhedrons in $\mathbb{R}^n$, such that two conditions are satisfied:

(1) If $Q \in \mathcal{S}$, and F is a face of Q, then $F \in \mathcal{S}$;

(2) If $Q_1, Q_2 \in \mathcal{S}$, then $Q_1 \cap Q_2$ is a face of Q_1 and Q_2 or $Q_1 \cap Q_2 = \emptyset$.

The subset $|\mathcal{S}| := \cup_{Q\in\mathcal{S}} Q$ of $\mathbb{R}^n$ equipped with the induced topology is called the underlying space of $\mathcal{S}$. The d-skeleton of $\mathcal{S}$ is the union of the faces whose dimension is at most d.

A dyadic complex is a polyhedral complex consisting of closed dyadic cubes. Let us refer to [21, 22] for the definition of dyadic cubes.

Let $\Omega \subset \mathbb{R}^n$ be an open subset, $0 < M < +\infty$, $0 < \delta \leq +\infty$, $\ell \in \mathbb{N}$, $0 \leq \ell \leq n$. For any map $f : \Omega \to \Omega$, we set

$$W_f = \{x \in \Omega \mid f(x) \neq x\}.$$

A δ-deformation is a family of maps $\{\varphi_t\}_{0\leq t\leq 1}$ from Ω into itself, which satisfy that φ_1 is Lipschitz and $\varphi_0 = \mathrm{id}_\Omega$, the function

$$[0,1] \times \Omega \to \Omega, (t,x) \mapsto \varphi_t(x)$$

is continuous, $\widehat{W}$ is relatively compact in Ω and $\operatorname{diam}(\widehat{W}) < \delta$, where

$$\widehat{W} = \bigcup_{t\in[0,1]} \left(W_{\varphi_t} \cup \varphi_t(W_{\varphi_t})\right). \tag{3.1.3}$$

Definition 3.4. Let E be a relatively closed set in Ω. We say that E is an (Ω, M, δ)-quasiminimal set of dimension ℓ if, $\mathcal{H}^\ell(E \cap B) < +\infty$ for every closed ball $B \subset \Omega$, and

$$\mathcal{H}^\ell(E \cap W_{\varphi_1}) \leq M\mathcal{H}^\ell(\varphi_1(E \cap W_{\varphi_1}))$$

for every δ-deformation $\{\varphi_t\}_{0\leq t\leq 1}$.

We denote by $QM(\Omega, M, \delta, \mathcal{H}^\ell)$ the collection of all (Ω, M, δ)-quasiminimal sets of dimension ℓ.

We note that, for any open set $\Omega' \subset \Omega$, any positive numbers $\delta' \leq \delta$, and any $M' \geq M$, if $E \in QM(\Omega, M, \delta, \mathcal{H}^\ell)$, then $E \cap \Omega' \in QM(\Omega', M', \delta', \mathcal{H}^\ell)$.

Definition 3.5. Let Ω, M, δ and ℓ be as above, and let $\varepsilon \in [0,1)$ be given. We shall denote by $GQM(\Omega, M, \delta, \varepsilon, \mathcal{H}^\ell)$ the set of closed subsets E of Ω such that $\mathcal{H}^\ell(E \cap B) < +\infty$ for every closed ball $B \subset \Omega$, and

$$\mathcal{H}^\ell(E \cap W_{\varphi_1}) \leq M\mathcal{H}^\ell(\varphi_1(E \cap W_{\varphi_1})) + \varepsilon\delta^\ell$$

for every δ-deformation $\{\varphi_t\}_{0\leq t\leq 1}$.

Definition 3.6. Let Ω be an open subset of $\mathbb{R}^n$. A relatively closed set $E \subset \Omega$ is said to be locally Ahlfors-regular of dimension d if there is a constant $C > 0$ and $r_0 > 0$ such that

$$C^{-1}r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq Cr^d$$

for all $0 < r < r_0$ with $B(x, 2r) \subset \Omega$.

Lemma 3.7. *Let E be a d-rectifiable subset of $\mathbb{R}^n$. If E is a local Ahlfors-regular and $\mathcal{H}^d(E) < +\infty$, then for $\mathcal{H}^d$-a.e. $x \in E$, E has a true tangent plane at x, i.e. there exists a d-plane π such that for any $\varepsilon > 0$, there is a $r_\varepsilon > 0$ such that*

$$E \cap B(x,r) \subset \mathcal{C}(x,\pi,r,\varepsilon), \quad \text{for } 0 < r < r_\varepsilon,$$

where

$$\mathcal{C}(x,\pi,r,\varepsilon) = \{y \in B(x,r) \mid \mathrm{dist}(y,\pi) \leq \varepsilon\,|y-x|\}.$$

Proof. Since E is rectifiable, by Theorem 15.11 in [29], for $\mathcal{H}^d$-a.e. $x \in E$, E has an approximate tangent plane π at x, i.e.

$$\limsup_{\rho\to 0} \frac{\mathcal{H}^d(E \cap B(x,\rho))}{\rho^d} > 0,$$

and there exists a d-plane π such that for all $\varepsilon > 0$,

$$\lim_{\rho\to 0} \frac{\mathcal{H}^d(E \cap B(x,\rho) \setminus \mathcal{C}(x,\pi,\rho,\varepsilon))}{\rho^d} = 0. \tag{3.1.4}$$

We will show that π is a true tangent plane. Suppose not, that is, there exists an $\varepsilon > 0$ such that for all $\rho > 0$, $E \cap B(x,\rho) \setminus \mathcal{C}(x,\pi,\rho,\varepsilon) \neq \emptyset$. We take a sequence of points $y_n \in E \setminus \mathcal{C}(x,\pi,\rho,\varepsilon)$ with $|y_n - x| \to 0$, we put $\rho_n = 2\,|y_n - x|$, then

$$B(x,\rho_n) \setminus \mathcal{C}\left(x,\pi,\rho_n,\frac{\varepsilon}{2}\right) \supset B\left(y_n, \frac{\varepsilon\rho_n}{4}\right)$$

and

$$\rho_n^{-d}\mathcal{H}^d\left(E\cap B(x,\rho_n)\setminus \mathcal{C}\left(x,\pi,\rho_n,\frac{\varepsilon}{2}\right)\right)\geq \rho_n^{-d}\mathcal{H}^d\left(E\cap B\left(y_n,\frac{\varepsilon\rho_n}{4}\right)\right)$$
$$\geq C^{-1}\left(\frac{\varepsilon}{4}\right)^d,$$

this is in contradiction with (3.1.4), so we proved the lemma. □

Let $\{E_k\}$ be a sequence of closed sets in Ω, and E a closed set of Ω. We say that E_k converges to E if

$$\lim_{k\to\infty} d_K(E,E_k)=0 \text{ for every compact set } K\subset\Omega,$$

where

$$d_K(E,E_k)=\left\{\sup_{x\in E\cap K}\operatorname{dist}(x,E_k),\ \sup_{x\in E_k\cap K}\operatorname{dist}(x,E)\right\}.$$

For any set $E\subset\mathbb{R}^n$, we set

$$E^*=\{x\in E\mid \mathcal{H}^d(E\cap B(x,r))>0,\ \forall r>0\};$$

we call E^* the core of E. We see from [8, p.78] that

$$\mathcal{H}^d(E\setminus E^*)=0,$$

and that E^* is also (generalized) quasiminimal when E is (generalized) quasiminimal with same constant. So we always assume that $E=E^*$.

Given a sequence of quasiminimal sets with the same constant and a generalized elliptic integrand, we will prove the following lower semicontinuity properties.

Theorem 3.8. *Let $\Omega\subset\mathbb{R}^n$ be an open set. Let $(E_k)_{k\geq1}$ be a sequence of quasiminimal sets in $GQM(\Omega,M,\delta,\varepsilon_0,\mathcal{H}^d)$ such that E_k converges to E. We assume that $\varepsilon_0\in(0,1)$ small enough, depending on d and M. Then for any generalized elliptic integrand F,*

$$J_F(E)\leq\liminf_{k\to+\infty}J_F(E_k).$$

Proof. We may suppose that

$$\liminf_{k\to+\infty}\mathcal{H}^d(E_k)<+\infty;$$

otherwise we have nothing to prove. By Theorem 3.4 in [7], we have that

$$\mathcal{H}^d(E)\leq\liminf_{k\to+\infty}\mathcal{H}^d(E_k)<+\infty.$$

We take $0 < \varepsilon < 1 - \varepsilon_0$, $\varepsilon' > 0$ and $\rho \in (0,1)$ such that

$$(1 + C(M,d))M^2 3^d \omega_d \varepsilon < 1 - \frac{2\varepsilon_0}{\omega_d},\ \varepsilon' < \frac{\varepsilon}{8} \text{ and } 1 - (1-\rho)^d < \frac{\varepsilon}{2},$$

where ω_d denote the Hausdorff measure of d-dimensional unit ball. Here we can suppose that $\varepsilon_0 < \frac{\omega_d}{2}$ because we have already assumed that ε_0 small enough. The constant $C(n,d)$ will be chosen later.

Applying Theorem 4.1 in [7], we get that $E \in QM(\Omega, M, \delta, \mathcal{H}^d)$, hence rectifiable (see [4]), then by Theorem 17.6 in [29], for $\mathcal{H}^d$-a.e. $x \in E$,

$$\lim_{r \to 0} \frac{\mathcal{H}^d(E \cap B(x,r))}{\omega_d r^d} = 1.$$

So we can find a set $E' \subset E$ with $\mathcal{H}^d(E \setminus E') = 0$ such that for any $x \in E'$ there exists $r'(\varepsilon', x) > 0$,

$$(1 - \varepsilon')\omega_d r^d \le \mathcal{H}^d(E \cap B(x,r)) \le (1 + \varepsilon')\omega_d r^d,$$

for all $0 < r < r'(\varepsilon', x)$.

Then

$$\begin{aligned}
\mathcal{H}^d(E \cap B(x,r) \setminus B(x,(1-\rho)r)) &\le (1+\varepsilon')\omega_d r^d - (1-\varepsilon')\omega_d(1-\rho)^d r^d \\
&= \frac{(1+\varepsilon') - (1-\varepsilon')(1-\rho)^d}{1-\varepsilon'}(1-\varepsilon')\omega_d r^d \\
&\le \left(\frac{2\varepsilon'}{1-\varepsilon'} + \left(1 - (1-\rho)^d\right)\right)\mathcal{H}^d(E \cap B(x,r)) \\
&\le \varepsilon \mathcal{H}^d(E \cap B(x,r)).
\end{aligned}$$

Since E is quasiminimal, by Proposition 4.1 in [14], we know that E is local Ahlfors regular, since E is rectifiable and $\mathcal{H}^d(E) < +\infty$, by lemma 3.7, we have that for $\mathcal{H}^d$-a.e. $x \in E$, E has a tangent space $T_x E$ at x, so we can find $E'' \subset E'$ with $\mathcal{H}^d(E' \setminus E'') = 0$ such that for all $\varepsilon'' > 0$ and for all $x \in E''$ there exists $r''(\varepsilon'', x) > 0$ such that for all $0 < r < r''(\varepsilon'', x)$,

$$E \cap B(x,r) \subset \mathcal{C}(x, r, \varepsilon''),$$

where

$$\mathcal{C}(x,r,\varepsilon'') = \left\{ y \in \overline{B(x,r)} \,\middle|\, \operatorname{dist}(y, T_x E) \le \varepsilon'' |x - y| \right\}.$$

We consider the function $\psi_{\rho,r} : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_{\rho,r}(t) = \begin{cases} 0, & t \le (1-\rho)r \\ \frac{3}{\rho r}\left(t - (1-\rho)r\right), & (1-\rho)r < t \le (1 - \frac{2\rho}{3})r \\ 1, & (1 - \frac{2\rho}{3})r < t \le (1 - \frac{\rho}{3})r \\ -\frac{3}{\rho r}(t - r), & (1 - \frac{\rho}{3})r < t \le r \\ 0, & t > r, \end{cases}$$

It is easy to see that $\psi_{\rho,r}$ is a Lipschitz map with Lipschitz constant $\frac{3}{\rho r}$.

We take the Lipschitz map $\varphi_{x,\rho,r} : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\varphi_{x,\rho,r}(y) = \psi_{\rho,r}(|y-x|)\Pi(y) + (1-\psi_{\rho,r}(|y-x|))y,$$

where we denote by $\Pi : \mathbb{R}^n \to T_xE$ the orthogonal projection. It is easy to check that

$$\varphi_{x,\rho,r}|_{B(x,(1-\rho)r)} = \mathrm{id}_{B(x,(1-\rho)r)}$$

and

$$\varphi_{x,\rho,r}|_{B(x,r)^c} = \mathrm{id}_{B(x,r)^c}.$$

Let ε'' and h be such that $\varepsilon'' < \frac{\rho}{3}$ and $0 < \varepsilon'' < h < \frac{\rho}{3}$, and put

$$A_h = \left\{ y \in \overline{B(x,r)} \;\middle|\; \mathrm{dist}(y, T_xE) \le hr \right\},$$

then $\mathcal{C}(x,r;\varepsilon'') \subset A_h$. We will show that

$$\mathrm{Lip}\left(\varphi_{x,\rho,r}|_{A_h}\right) \le 2 + \frac{3h}{\rho}.$$

We set

$$\Pi^\perp(y) = y - \Pi(y),\ y \in \mathbb{R}^n,$$

then

$$\left|\Pi^\perp(y)\right| \le hr,\ \forall y \in A_h.$$

For any $y_1, y_2 \in A_h$,

$$\begin{aligned}
\varphi_{x,\rho,r}(y_1) - \varphi_{x,\rho,r}(y_2) &= y_1 - y_2 + \psi_{\rho,r}(|y_1-x|)\Pi^\perp(y_1) - \psi_{\rho,r}(|y_2-x|)\Pi^\perp(y_2) \\
&= (y_1-y_2) + \psi_{\rho,r}(|y_1-x|)\left(\Pi^\perp(y_1) - \Pi^\perp(y_2)\right) \\
&\quad + \left(\psi_{\rho,r}(|y_1-x|) - \psi_{\rho,r}(|y_2-x|)\right)\Pi^\perp(y_2),
\end{aligned}$$

thus

$$\begin{aligned}
|\varphi_{x,\rho,r}(y_1) - \varphi_{x,\rho,r}(y_2)| &\le |y_1-y_2| + |y_1-y_2| + \frac{3}{\rho r}\left||y_1| - |y_2|\right| rh \\
&\le \left(2 + \frac{3h}{\rho}\right)|y_1-y_2|,
\end{aligned}$$

and we get that

$$\mathrm{Lip}\left(\varphi_{x,\rho,r}|_{A_h}\right) \le 2 + \frac{3h}{\rho}.$$

Since $E_k \to E$ in Ω, and $\overline{B(x,r)} \subset \Omega$ and

$$E \cap \overline{B(x,r)} \subset \mathcal{C}(x,r,\varepsilon'') \subset A_h,$$

there exist a number k_h such that for $k \geq k_h$,

$$E_k \cap \overline{B(x,r)} \subset A_h.$$

Since

$$\varphi_{x,\rho,r}|_{B(x,r)^c} = \mathrm{id}_{B(x,r)^c}$$

and

$$\varphi_{x,\rho,r}(B(x,r)) \subset B(x,r),$$

we have that

$$\varphi_{x,\rho,r}(E_k \cap B(x,r)) = \varphi_{x,\rho,r}(E_k) \cap B(x,r).$$

We put $r' = \left(1 - \frac{\rho}{3}\right) r$, $r'' = \left(1 - \frac{2\rho}{3}\right) r$, $r''' = (1-\rho)r$, $\pi = T_x E$, and put

$$D_{\pi,r''} = \overline{B\left(x, r''\right)} \cap \pi$$

and

$$S_{k,r''} = \varphi_{x,\rho,r}(E_k) \cap \overline{B\left(x, r''\right)}.$$

Note that

$$\partial B\left(x, r'\right) \cap \pi \subset \varphi_{x,\rho,r}(E_k)$$

and

$$\varphi_{x,\rho,r}(E_k) \cap B\left(x, r'\right) \subset B\left(x, r''\right) \cup \left(\left(B(x,r') \setminus B(x,r'')\right) \cap \pi\right).$$

We will show that for any Lipschitz mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ which is identity on $\partial D_{\pi,r''}$ cannot map $S_{k,r''}$ into $\partial D_{\pi,r''}$. Suppose not, that is, there is a Lipschitz map $\varphi_k : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi_k|_{\partial D_{\pi,r''}} = \mathrm{id}_{\partial D_{\pi,r''}}$$

and

$$\varphi_k(S_{k,r''}) \subset \partial D_{\pi,r''}.$$

We consider the map

$$\tilde{\phi}_k : B(x,\eta)^c \cup [(B(x,\eta) \setminus B(x,r'')) \cap \pi] \cup B(x,r'') \to \mathbb{R}^n$$

defined by

$$\tilde{\phi}_k(x) = \begin{cases} x, & x \in B(x,\eta)^c \cup [(B(x,\eta) \setminus B(x,r'')) \cap \pi] \\ \varphi_k(x), & x \in B(x,r''), \end{cases}$$

where η is a number such that $r'' < \eta < r'$. It is easy check that $\tilde{\phi}_k$ is a Lipschitz map, by Kirszbraun's theorem, see for example [19, 2.10.43 Kirszbraun's theorem], we can get a Lipschitz map $\phi_k : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\phi_k|_{B(x,r'')} = \varphi_k|_{B(x,r'')}$$

and

$$\phi_k|_{B(x,\eta)^c} = \mathrm{id}_{B(x,\eta)^c},$$

and

$$\phi_k|_{(B(x,\eta)\setminus B(x,r''))\cap\pi} = \mathrm{id}_{(B(x,\eta)\setminus B(x,r''))\cap\pi}.$$

Recalling the construction of $\psi_{\rho,r}$, we have that

$$0 \le \psi_{\rho,r} \le 1,\ \psi_{\rho,r}|_{[0,r''']} = 0,\ \psi_{\rho,r}|_{[r'',r']} = 1,\ \psi_{\rho,r}|_{[r,+\infty)} = 0;$$

and then recalling the construction of $\varphi_{x,\rho,r}$, we have that

$$\varphi_{x,\rho,r}|_{B(x,r)^c} = \mathrm{id}_{B(x,r)^c},\ \varphi_{x,\rho,r}(B(x,r)) \subset B(x,r),\ \varphi_{x,\rho,r}(B(x,r'')) \subset B(x,r'').$$

and

$$\varphi_{x,\rho,r}|_{B(x,r')\setminus B(x,r'')} = \Pi_{B(x,r')\setminus B(x,r'')},\ \varphi_{x,\rho,r}|_{B(x,r''')} = \mathrm{id}_{B(x,r''')},$$

where Π is the orthogonal projection onto the plane π. Thus $\varphi_{x,\rho,r}$ coincide with the orthogonal projection Π on the annulus $\overline{B(x,r')} \setminus B(x,r'')$. Note that $\varepsilon'' < \frac{\rho}{3}$, we get that

$$1 - \frac{2\rho}{3} < \left(1 - \frac{\rho}{3}\right)\sqrt{1 - \varepsilon''^2}.$$

We now let the number η be such that $r'' < \eta < (\sqrt{1 - \varepsilon''^2})r'$. Since $E_k \cap B(x,r) \subset \mathcal{C}(x,r,\varepsilon'')$, we have that

$$\varphi_{x,\rho,r}(E_k \cap B(x,r'')) \subset B(x,r''),\ \varphi_{x,\rho,r}(E_k \cap (B(x,r') \setminus B(x,r'')) \subset \pi,$$

and

$$\varphi_{x,\rho,r}(E_k \cap (B(x,r) \setminus B(x,r'))) \subset B(x,r) \setminus B(x,\eta),$$

thus

$$\varphi_{x,\rho,r}(E_k) \cap (B(x,\eta) \setminus B(x,r'')) \subset (B(x,\eta) \setminus B(x,r'')) \cap \pi,$$

and

$$\phi_k\left(\varphi_{x,\rho,r}(E_k) \cap B(x,\eta) \setminus B(x,r'')\right) \subset (B(x,\eta) \setminus B(x,r'')) \cap \pi.$$

Thus we get that

$$\begin{aligned}\phi_k(\varphi_{x,\rho,r}(E_k\cap B(x,r))) &= \phi_k(\varphi_{x,\rho,r}(E_k)\cap B(x,r))\\ &\subset \varphi_{x,\rho,r}(E_k)\cap (B(x,r)\setminus B(x,r''))\\ &\subset \varphi_{x,\rho,r}(E_k\cap B(x,r)\setminus B(x,r'')).\end{aligned}$$

Since $\mathcal{H}^d(E)<\infty$, we have that $\mathcal{H}^d(E\cap\partial B(x,r))=0$ for almost every where $r\in(0,r''(\varepsilon'',x))$, if we take any $r\in(0,r''(\varepsilon'',x))$ with $\mathcal{H}^d(E\cap\partial B(x,r))=0$ and $r<\delta$, by using Lemma 3.12 in [8], we get that

$$\limsup_{k\to+\infty}\mathcal{H}^d\left(E_k\cap\overline{B(x,r)}\setminus B(x,r''')\right)\leq(1+C(M,d)\varepsilon_0)M\mathcal{H}^d(E\cap\overline{B(x,r)}\setminus B(x,r''')).$$

We put $M'=(1+C(M,d)\varepsilon_0)M$. Then we have the following inequality:

$$\begin{aligned}\mathcal{H}^d\left(E\cap\overline{B(x,r)}\right) &= \mathcal{H}^d(E\cap B(x,r))\\ &\leq \liminf_{k\to+\infty}\mathcal{H}^d(E_k\cap B(x,r))\\ &\leq \liminf_{k\to+\infty}M\mathcal{H}^d\left(\phi_k\circ\varphi_{x,\rho,r}(E_k\cap B(x,r))\right)+\varepsilon_0r^d\\ &\leq \liminf_{k\to+\infty}M\mathcal{H}^d\left(\varphi_{x,\rho,r}(E_k\cap B(x,r)\setminus B(x,r'''))\right)+\varepsilon_0r^d\\ &\leq \liminf_{k\to+\infty}M\mathcal{H}^d\left(\varphi_{x,\rho,r}(E_k\cap\overline{B(x,r)}\setminus B(x,r'''))\right)+\varepsilon_0r^d\\ &\leq \liminf_{k\to+\infty}M\left(2+\frac{3h}{\rho}\right)^d\mathcal{H}^d\left(E_k\cap\overline{B(x,r)}\setminus B(x,r''')\right)+\varepsilon_0r^d\\ &\leq M\left(2+\frac{3h}{\rho}\right)^d\limsup_{k\to+\infty}\mathcal{H}^d\left(E_k\cap\overline{B(x,r)}\setminus B(x,r''')\right)+\varepsilon_0r^d\\ &\leq M\left(2+\frac{3h}{\rho}\right)^d\cdot M'\mathcal{H}^d\left(E\cap\overline{B(x,r)}\setminus B(x,r''')\right)+\varepsilon_0r^d\\ &\leq MM'\left(2+\frac{3h}{\rho}\right)^d\varepsilon\mathcal{H}^d\left(E\cap\overline{B(x,r)}\right)+\varepsilon_0r^d\\ &\leq MM'3^d\varepsilon\mathcal{H}^d(E\cap B(x,r))+\varepsilon_0r^d\\ &\leq \left(MM'3^d\varepsilon+\frac{2\varepsilon_0}{\omega_d}\right)\mathcal{H}^d(E\cap B(x,r)).\end{aligned}$$

This is a contradiction since

$$MM'3^d\varepsilon+\frac{2\varepsilon_0}{\omega_d}=(1+C(M,d)\varepsilon_0)M^23^d\varepsilon+\frac{2\varepsilon_0}{\omega_d}<1$$

and

$$\mathcal{H}^d\left(E\cap B(x,r)\right)>0.$$

Since F is a generalized elliptic integrand, by Lemma 3.2, we have that

$$J_F(D_{\pi,r''}) \leq J_F(S_{k,r''}) + h(r'')(r'')^d.$$

Since E is a d-rectifiable set and $\mathcal{H}^d(E) < +\infty$, the function $f : E \to G(n,d)$ defined by $f(x) = T_xE$ is $\mathcal{H}^d$-measurable. By Lusin's theorem, see for example [19, 2.3.5. Lusin's theorem], we can find a closed set $N \subset E$ with $\mathcal{H}^d(E \setminus N) < \varepsilon$ such that f restricted to N is continuous. We put $E''' = (E'' \cap N)$, then $E''' \subset E$ and

$$\mathcal{H}^d(E \setminus E''') < \varepsilon,$$

by Lemma 15.18 in [29], we have that for $\mathcal{H}^d$-a.e. $x \in E'''$,

$$T_xE''' = T_xN = T_xE.$$

The map $\tilde{f} : E''' \to \mathbb{R}^n \times G(n,d)$ given by $\tilde{f}(x) = (x, T_xE)$ is continuous. Since F is continuous, thus the function $F \circ \tilde{f} : E''' \to \mathbb{R}$ is continuous, for any $x \in E'''$, we can find $r(\varepsilon, x) > 0$ such that

$$(1-\varepsilon)F(x, T_xE) \leq F(y, T_yE) \leq (1+\varepsilon)F(x, T_xE),$$

for any $y \in E''' \cap B(x, r(\varepsilon, x))$. Thus, for all $0 < r < r(\varepsilon, x)$,

$$(1-\varepsilon)J_F(T_xE \cap B(x,r)) \leq J_F(E''' \cap B(x,r)) \leq (1+\varepsilon)J_F(T_xE \cap B(x,r)).$$

For any $x \in \mathbb{R}^n$, there exists $r'''(\varepsilon) > 0$ such that $h(r) < \varepsilon$ for all $0 < r < r'''(\varepsilon)$. We put

$$r(x) = \min\{r(\varepsilon, x), r'(\varepsilon', x), r''(\varepsilon'', x), r'''(\varepsilon), \delta\}, \text{ for } x \in E'''.$$

Then

$$\left\{B(x,r) \;\middle|\; x \in E''', 0 < r < r(x), \mathcal{H}^d(E \cap \partial B(x,r)) = 0\right\}$$

is a Vitali covering of E''', so we can find a countable family of balls $(B_i)_{i \in J}$ such that

$$\mathcal{H}^d\left(E''' \setminus \bigcup_{i \in J} B_i\right) = 0.$$

We choose a finite set $I \subset J$ such that

$$\mathcal{H}^d\left(E''' \setminus \bigcup_{i \in I} B_i\right) < \varepsilon.$$

So we get that

$$J_F(E''') = \sum_{i\in J} J_F(E''' \cap B_i) \le \sum_{i\in I} J_F(E''' \cap B_i) + (\sup F)\varepsilon.$$

We denote $B_i = B(x_i, r_i)$, for $i \in I$, and put

$$\varphi = \prod_{i\in I} \varphi_{x_i,\rho,r_i}.$$

By definition, we get $\varphi|_{B_i} = \varphi_{x_i,\rho,r_i}|_{B_i}$. Thus $\varphi|_{B(x_i,r_i''')} = \mathrm{id}_{B(x_i,r_i''')}$,

$$\varphi(E_k) \cap B(x_i, r_i'') \setminus B(x_i, r_i''') \subset \varphi(E_k \cap B(x_i, r_i) \setminus B(x_i, r_i''')),$$

and

$$\pi_i \cap B = \pi_i \cap \left((B(x_i, r_i) \setminus B(x_i, r_i'')) \cup (B(x_i, r_i'') \setminus B(x_i, r_i''')) \cup (B(x_i, r_i'''))\right),$$

and $\varphi|_{A_h} \le 3$. We get that

$$\mathcal{H}^d(\varphi(E_k \cap B(x_i, r_i) \setminus B(x_i, r_i'''))) \le 3^d \mathcal{H}^d(E_k \cap B(x_i, r_i) \setminus B(x_i, r_i''')),$$

for $k \ge k_h$.

We denote $a = \sup\{F(x, \pi) \mid x \in \mathbb{R}^n, \pi \in G(n, d)\}$. Then for any $i \in I$,

$$\begin{aligned}
J_F(E''' \cap B_i) &\le (1+\varepsilon) J_F(\pi_i \cap B_i) \\
&\le (1+\varepsilon)\Big(J_F\left(\pi_i \cap B(x_i, r_i) \setminus B(x_i, r_i'')\right) + J_F(\pi_i \cap B(x_i, r_i''))\Big) \\
&\le (1+\varepsilon)\Big(J_F(S_{k,r''}) + \varepsilon(x, r_i'')(r_i'')^d + (\sup F)(r_i^d - (r_i'')^d)\Big) \\
&\le (1+\varepsilon) J_F(S_{k,r''}) + 2\varepsilon (r_i'')^d + 2(\sup F)\left((r_i^d - (r_i'')^d)\right) \\
&\le (1+\varepsilon) J_F(E_k \cap B(x_i, r_i''')) \\
&\quad + (1+\varepsilon) J_F(\varphi(E_k \cap B(x_i, r_i) \setminus B(x_i, r_i'''))) \\
&\quad + \left(2\varepsilon + 2a\left(1 - \left(1 - \frac{2\rho}{3}\right)^d\right)\right) r_i^d \\
&\le (1+\varepsilon) J_F(E_k \cap B(x_i, r_i''')) \\
&\quad + 2a3^d \mathcal{H}^d(E_k \cap B(x_i, r_i) \setminus B(x_i, r_i''')) \\
&\quad + \left(2\varepsilon + 2a \cdot \frac{\varepsilon}{2}\right) \frac{2}{\omega_d} \mathcal{H}^d(E \cap B(x_i, r_i)).
\end{aligned}$$

Hence

$$\begin{aligned}
J_F(E''') &\leq \sum_{i\in I} J_F(E''' \cap B_i) + a\varepsilon \\
&\leq (1+\varepsilon)J_F(E_k) + \sum_{i\in I}(2\varepsilon + a\varepsilon)\,\frac{2}{\omega_d}\mathcal{H}^d(E\cap B_i) + a\varepsilon \\
&\quad + 2a3^d\sum_{i\in I}\mathcal{H}^d\left(E_k \cap \overline{B(x_i,r_i)} \setminus B(x_i,r_i''')\right),
\end{aligned}$$

thus

$$\begin{aligned}
J_F(E''') &\leq \liminf_{k\to+\infty}(1+\varepsilon)J_F(E_k) + (2\varepsilon + a\varepsilon)\,\frac{2}{\omega_d}\mathcal{H}^d(E) + a\varepsilon \\
&\quad + 2a3^d \liminf_{k\to+\infty}\sum_{i\in I}\mathcal{H}^d\left(E_k \cap \overline{B(x_i,r_i)} \setminus B(x_i,r_i''')\right) \\
&\leq \liminf_{k\to+\infty}(1+\varepsilon)J_F(E_k) + (2\varepsilon + a\varepsilon)\frac{2}{\omega_d}\mathcal{H}^d(E) + a\varepsilon \\
&\quad + 2a3^d(1+C(M,d)\varepsilon_0)M\sum_{i\in I}\mathcal{H}^d\left(E \cap \overline{B(x_i,r_i)} \setminus B(x_i,r_i''')\right) \\
&\leq \liminf_{k\to+\infty}(1+\varepsilon)J_F(E_k) + (2\varepsilon + a\varepsilon)\frac{2}{\omega_d}\mathcal{H}^d(E) + a\varepsilon \\
&\quad + 2a3^d(1+C(M,d)\varepsilon_0)M\varepsilon\mathcal{H}^d(E),
\end{aligned}$$

and

$$\begin{aligned}
J_F(E) &= J_F(E''') + J_F(E\setminus E''') \\
&\leq J_F(E''') + a\mathcal{H}^d(E\setminus E''') \\
&\leq (1+\varepsilon)\liminf_{k\to+\infty} J_F(E_k) + 2a\varepsilon \\
&\quad + \left(\frac{4+2a}{\omega_d} + 2a3^d(1+C(M,d)\varepsilon_0)M\right)\mathcal{H}^d(E)\varepsilon.
\end{aligned}$$

We can let ε tend to 0, we get that

$$J_F(E) \leq \liminf_{k\to+\infty} J_F(E_k).$$

$\square$

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that there exists constant $c, C > 0$ such that for all $x \in \mathbb{R}^n$

$$0 < c \leq g(x) \leq C < +\infty.$$

As I mentioned before, the integrand F defined by

$$F(x, \pi) = g(x), \forall x \in \mathbb{R}^n, \ \pi \in G(n, d)$$

is a generalized elliptic integrand.

We will give some more examples of generalized elliptic integrand. For simplicity, we suppose that $n = 2, d = 1$. For any periodic continuous function $\lambda : \mathbb{R} \to \mathbb{R}$ with period π, λ can be regard as a contionuos function on the Grassmannian $G(2, 1)$. If λ is positive, i.e. $\lambda(\theta) > 0$ for all $\theta \in \mathbb{R}$, then the function $F : \mathbb{R}^2 \times G(2, 1) \to \mathbb{R}$ defined by

$$F(x, \theta) = \lambda(\theta) \text{ for any } x \in \mathbb{R}^2 \text{ and any } \theta \in G(2, 1)$$

is an integrand.

Example 3.9. Let a and b be two real number such that $a > |b|$. Then the integrand F defined by $F(x, \theta) = a + b \cos\theta$ is a generalized elliptic integrand. Let ABC be a triangle. We denote by α, β and γ the angle between the x-axis

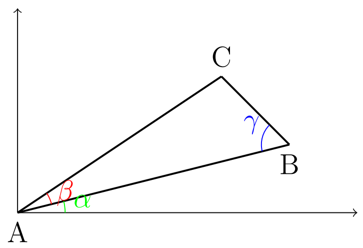


Figure 3.1: triangle

and the segment AB, the angle BAC and the angle ABC respectively. Then

$$J_F(AB) = (a + b\cos\alpha)|AB|, \ J_F(AC) = (a + b\cos(\alpha + \beta))|AC|$$

and

$$J_F(BC) = (a + b\cos(\alpha - \gamma))|BC|.$$

We put $\Lambda = J_F(AC) + J_F(BC) - J_F(AB)$. Since $|AB| = |AC|\cos\beta + |BC|\cos\gamma$ and $|AC|\sin\beta = |BC|\sin\gamma$, we get that

$$\begin{aligned}
\Lambda &= (a + b\cos(\alpha + \beta))|AC| - (a + b\cos\alpha)|AC|\cos\beta \\
&\quad + (a + b\cos(\alpha - \gamma))|BC| - (a + b\cos\alpha)|BC|\cos\gamma \\
&= a(1 - \cos\beta)|AC| + a(1 - \cos\gamma)|BC| \\
&\geq 0.
\end{aligned}$$

Thus

$$J_F(AB) \leq J_F(AC) + J_F(BC), \tag{3.1.5}$$

and F is a generalized elliptic integrand. $\square$

Example 3.10. For any periodic C^2 function $\lambda : \mathbb{R} \to \mathbb{R}$ with period π, there exists a constant C_0 such that for any $C > C_0$ the integrand F defined by $F(x, \theta) = C + \lambda(\theta)$ is a generalized elliptic integrand.

For any $\alpha \in [0, \pi]$, we consider the function $f_\alpha : [-\pi, \pi] \to \mathbb{R}$ defined by

$$f_\alpha(\theta) = \begin{cases} \dfrac{\lambda'(\alpha+\theta)\sin\theta - \lambda(\alpha+\theta)\cos\theta + \lambda(\alpha)}{1-\cos\beta}, & \text{if } \theta \in [\pi, 0) \cup (0, \pi]; \\ \lambda(\alpha) + \lambda''(\alpha), & \text{if } \theta = 0. \end{cases}$$

We will show that f_α is continuous. Indeed, for any $\theta \in [-\pi, \pi] \setminus \{0\}$, we have that

$$f_\alpha(\theta) = \lambda(\alpha+\theta) + \frac{(\lambda'(\alpha+\theta) - \lambda'(\alpha))\sin\theta}{1-\cos\theta} - \frac{\lambda(\alpha+\theta) - \lambda(\alpha) - \lambda'(\alpha)\sin\theta}{1-\cos\theta}.$$

Thus

$$\lim_{\theta \to 0} f_\alpha(\theta) = \lambda(\alpha) + \lambda''(\alpha),$$

and f_α is continuous. We get that the function $f : [-\pi, \pi] \times [-\pi, \pi] \to \mathbb{R}$ defined by $f(\alpha, \theta) = f_\alpha(\theta)$ for any $\alpha, \theta \in [-\pi, \pi]$ is continuous. We take

$$C_0 = \max\left\{ \sup_{\alpha,\theta \in [-\pi,\pi]} |f(\alpha, \theta)|, \sup_{\alpha \in [-\pi,\pi]} |\lambda(\alpha)| \right\}.$$

For any $C \geq C_0$, we consider the integrand F defined by $F(x, \theta) = C + \lambda(\theta)$. We let ABC be a triangle, denote by α, β and γ the angle between the x-axis

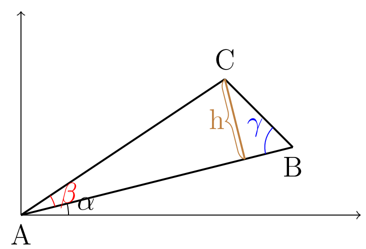


Figure 3.2: triangle

and the segment AB, the angle BAC and the angle ABC respectively. We put $h = \mathrm{dist}(C, AB)$. Then

$$J_F(AC) = (C + \lambda(\alpha+\beta))\frac{h}{\sin\beta}, \quad J_F(BC) = (C + \lambda(\gamma-\alpha))\frac{h}{\sin\gamma}$$

and

$$J_F(AB) = (h\cot\beta + h\cot\gamma)\,(C + \lambda(\alpha)).$$

We put $\Lambda = J_F(AC) + J_F(BC) - J_F(AB)$. Then

$$\begin{aligned}\Lambda = h&\frac{\lambda(\alpha+\beta) - \lambda(\alpha)\cos\beta + C(1-\cos\beta)}{\sin\beta}\\ &+ h\frac{\lambda(\alpha-\gamma) - \lambda(\alpha)\cos\gamma + C(1-\cos\gamma)}{\sin\gamma}\end{aligned}$$

For any $\alpha \in [-\pi, \pi]$, we consider the function $g_\alpha : (-\pi,\pi) \to \mathbb{R}$ defined by

$$g_\alpha(\beta) = \begin{cases} \dfrac{\lambda(\alpha+\beta) - \lambda(\alpha)\cos\beta + C(1-\cos\beta)}{\sin\beta}, & \text{if } \beta \in (-\pi,0)\cup(0,\pi);\\ \lambda'(\alpha), & \text{if } \beta = 0.\end{cases}$$

We will show that $g_\alpha \in C^1(-\pi,\pi)$ and that $g'_\alpha \geq 0$. Indeed, we have that

$$\lim_{\beta\to 0} g_\alpha(\beta) = \lim_{\beta\to 0}\left(\frac{\lambda(\alpha+\beta)-\lambda(\alpha)}{\sin\beta} + \frac{(C+\lambda(\alpha))(1-\cos\beta)}{\sin\beta}\right) = \lambda'(\alpha).$$

Thus g_α is continuous. Since

$$\frac{g_\alpha(\beta) - g_\alpha(0)}{\beta} = \frac{(\lambda(\alpha+\beta) - \lambda(\alpha) - \lambda'(\alpha)\sin\beta) + (C+\lambda(\alpha))(1-\cos\beta)}{\beta\sin\beta},$$

we get that

$$g'_\alpha(0) = \lim_{\beta\to 0}\frac{g_\alpha(\beta)-g_\alpha(0)}{\beta} = \frac{1}{2}(\lambda''(\alpha) + \lambda(\alpha) + C).$$

But for any $\beta \in (-\pi,0)\cup(0,\pi)$, we have that

$$g'_\alpha(\beta) = \frac{\lambda'(\alpha+\beta) - \lambda(\alpha+\beta)\cos\beta + \lambda(\alpha) + C(1-\cos\beta)}{\sin^2\beta} = \frac{f_\alpha(\beta)+C}{1+\cos\beta}.$$

And hence

$$\lim_{\beta\to 0} g'_\alpha(\beta) = \frac{1}{2}(\lambda''(\alpha)+\lambda(\alpha)+C) = g'_\alpha(0).$$

We get that $g'_\alpha \in C^1(-\pi,\pi)$. Since $|f_\alpha| \leq C_0 \leq C$, we get that $g'_\alpha \geq 0$.

Since $\beta, \gamma \in (0,\pi)$, we get that

$$\Lambda = g_\alpha(\beta) - g_\alpha(-\gamma) \geq 0.$$

Thus

$$J_F(AB) \leq J_F(AC) + J_F(BC),$$

and F is a generalized elliptic integeand.

□

Example 3.11. Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a positive periodic continuous function with period π, $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $0 < c \leq g \leq C < +\infty$ for some constant c and C. If the integrand F defined by $F(x,\theta) = \lambda(\theta)$ is a generalized elliptic integrand, then the integrand $\widetilde{F}$ defined by $\widetilde{F}(x,\theta) = g(x)\lambda(\theta)$ is a generalized elliptic integrand. $\square$

3.1.2 Polyhedral approximation

In this section we will adapt some results of [21, 22].

Proposition 3.12. *Suppose that $0 < d < n$ and that F is integrand. Then there is a positive constant $M > 0$ such that for all open bounded domain $U \subset \mathbb{R}^n$, for all closed set $E \subset U$ with finite Hausdorff measure, and all $\varepsilon > 0$, we can build a n-dimensional complex $\mathcal{S}$ and a Lipschitz map $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following properties:*

(1) $\phi|_{\mathbb{R}^n \setminus U} = \mathrm{id}_{\mathbb{R}^n \setminus U}$ *and* $\|\phi - \mathrm{id}\| \leq \varepsilon$;

(2) $\phi(E)$ *is contained in the union of d-skeleton of $\mathcal{S}$, and $|\mathcal{S}| \subset U$;*

(3) $J_F(\phi(E)) \leq (1+\varepsilon) J_F(E)$.

This is only a small improvement over Theorem 4.3.17 in [21] and Theorem 3 in [22], but the proof is almost same as that of V. Feuvrier in [21, 22].

The inequality $J_F(\phi(E)) \leq (1+\varepsilon)J_F(E)$ can be replaced by $J_{F,f}(\phi(E)) \leq (1+\varepsilon)J_{F,f}(E)$ with a function $f : \mathbb{R}^n \to \mathbb{R}^+$ which satisfies

$$f(x) \geq \sup_{\pi \in G(n,d)} F(x,\pi),$$

but the proof is same.

Proof. we decompose E as $E = E_{rec} \sqcup E_{irr}$, where E_{rec} is d-rectifiable and E_{irr} is d-irregular. For any $\varepsilon' > 0$, by Lemma 4 in [22], for $\mathcal{H}^d$ almost every $x \in E_{rec}$, we can find $r_{max}(x) > 0$, $\rho \in (0,1)$, $u > 0$ and $T_x E_{rec}$ such that for all $r \in (0, r_{max})$,

$$\mathcal{H}^d(\Pi_{H,\rho r,\mathcal{C}(x,r,u)}(E \cap B(x, r+r\rho) \setminus \mathcal{C}(x,r,u))) \leq \varepsilon' \mathcal{H}^d(E \cap B(x, r+r\rho)), \quad (3.1.6)$$

where H we denote the approximate tangent plane $T_x E_{rec}$.

Since E_{rec} is a d-rectifiable set and $\mathcal{H}^d(E_{rec}) < +\infty$, the function $g : E_{rec} \to G(n,d)$ defined by $g(x) = T_x E_{rec}$ is $\mathcal{H}^d$-measurable. By Lusin's theorem, see for example [19, 2.3.5. Lusin's theorem], we can find a closed set $E' \subset E_{rec}$ with $\mathcal{H}^d(E_{rec} \setminus E') < \varepsilon' \mathcal{H}^d(E)$ such that g restricted to F is continuous. Thus the function $E' \to E' \times G(n,d)$ defined by $x \mapsto (x, T_x E_{rec})$

is continuous. For any $x \in E'$, we can find $r'_{max}(x) > 0$ such that for all $y \in E' \cap B(x, r'_{max})$,

$$(1-\varepsilon')F(x, T_x E_{rec}) \leq F(y, T_y E_{rec}) \leq (1+\varepsilon')F(x, T_x E_{rec}). \tag{3.1.7}$$

We consider the function $\tilde{F} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\tilde{F}(x) = \sup_{\pi \in G(n,d)} F(x, \pi),$$

$\tilde{F}$ is continuous, thus for any $x \in \mathbb{R}^n$ there exists $r''_{\max} > 0$ such that

$$\forall y \in B(x, r''_{max}),\ (1-\varepsilon')\tilde{F}(x) \leq \tilde{F}(y) \leq (1+\varepsilon')\tilde{F}(x)$$

Since F is a continuous function and $G(n, d)$ is compact, for any $x \in \mathbb{R}^n$ we can find $r'''_{max} > 0$ such that for all $y \in B(x, r'''_{max})$ and $\pi \in G(n, d)$,

$$(1-\varepsilon')F(x, \pi) \leq F(y, \pi) \leq (1+\varepsilon')F(x, \pi).$$

Let $\mathcal{B}$ be the collection of closed balls with center in E' and radius at most

$$\min\left(\frac{r_{max}}{1+\rho}, r'_{max}, r''_{max}, r'''_{max}, \frac{\varepsilon}{2}\right).$$

By a Vitali covering lemma, we can find a countable many pairwise disjoint balls $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}$ such that

$$\mathcal{H}^d\left(E' \setminus \bigcup_{i=1}^{\infty} B_i\right) = 0.$$

We take an integer N such that

$$\mathcal{H}^d\left(E' \setminus \bigcup_{i=1}^{N} B_i\right) \leq \varepsilon' \mathcal{H}^d(E).$$

We suppose that $B_i = \overline{B(x_i, r'_i)}$ with $x_i \in E'$ and $r'_i \leq r_{max}(x_i)$. We put $r_i = \frac{r'_i}{1+\rho_i}$, $K_i = \mathcal{C}(x_i, r_i, u_i)$ and $H_i = T_{x_1} E_{rec}$, where ρ_i and u_i are the numbers ρ and u in (3.1.6).

We consider the map

$$\psi_0 = \prod_{i=1}^{N} \Pi_{H_i, r_i\rho_i, K_i},$$

it is γ-Lipschitz with

$$\gamma \leq \left(2 + \max_{1\leq i\leq N} \frac{u_i}{\rho_i}\right),$$

and by the construction, we know that ψ_0 is identity on $(\cup_{i=1}^N B_i)^c$.

Since ψ_0 is identity on $\cup_{1\leq i\leq N} B_i$, we have

$$J_F\left(\psi_0\left(E \setminus \bigcup_{i=1}^N B_i\right)\right) = J_F\left(E \setminus \bigcup_{i=1}^N B_i\right).$$

Let $b = \sup_{(x,\pi)} F(x,\pi)$ and $a = \inf_{(x,\pi)} F(x,\pi)$. Since ψ_0 is γ-Lipschitz and

$$\mathcal{H}^d(E \cap B_i \setminus K_i) \leq \varepsilon' \mathcal{H}^d(E \cap B_i),$$

we get that

$$J_F(\psi_0(E \cap B_i \setminus K_i)) \leq \gamma^d b \varepsilon' \mathcal{H}^d(E \cap B_i) \leq \frac{\gamma^d b \varepsilon'}{a} J_F(E \cap B_i).$$

and

$$\begin{aligned} J_F(\psi_0((E_{rec} \setminus E') \cap K_i)) &\leq \gamma^d b \mathcal{H}^d((E_{rec} \setminus E') \cap K_i) \\ &\leq \gamma^d b \mathcal{H}^d((E_{rec} \setminus E') \cap B_i). \end{aligned}$$

By (3.1.7), we get that

$$(1-\varepsilon')F(x_i, H_i)\mathcal{H}^d(E' \cap K_i) \leq J_F(E' \cap K_i) \leq (1+\varepsilon')F(x_i, H_i)\mathcal{H}^d(E' \cap K_i),$$

$$\begin{aligned} J_F(\psi_0(E' \cap K_i)) &\leq (1+\varepsilon')F(x_i, H_i)\mathcal{H}^d(\psi_0(E' \cap K_i)) \\ &\leq (1+\varepsilon')F(x_i, H_i)\mathcal{H}^d(E' \cap K_i) \\ &\leq \frac{1+\varepsilon'}{1-\varepsilon'} J_F(E' \cap K_i) \\ &\leq \frac{1+\varepsilon'}{1-\varepsilon'} J_F(E' \cap B_i). \end{aligned}$$

Since ψ_0 is the orthogonal projection to H_i in a neighborhood of K_i,

$$\psi_0(E_{irr} \cap K_i) \subset H_i \cap B_i,$$

$\psi_0(E_{irr} \cap K_i)$ is rectifiable,

$$\begin{aligned} J_F(\psi_0(E_{irr} \cap K_i)) &\leq (1+\varepsilon')F(x_i, H_i)\mathcal{H}^d(\psi_0(E_{irr} \cap K_i)) \\ &\leq (1+\varepsilon')F(x_i, H_i)\mathcal{H}^d(E_{irr} \cap K_i) \\ &\leq (1+\varepsilon')^2 J_F(E_{irr} \cap K_i). \end{aligned}$$

Put $S = \cup_{i=1}^{N} B_i$. Note that

$$E \cap K_i = ((E_{rec} \setminus E') \cap K_i) \cup (E' \cap K_i) \cup (E_{irr} \cap K_i),$$

we have that

$$\begin{aligned}
J_F(\psi_0(E)) &\leq J_F(\psi_0(E \setminus S)) + J_F(\psi_0(E \cap S)) \\
&\leq J_F(E \setminus S) + J_F(\psi_0(E \cap S)) \\
&\leq J_F(E \setminus S) + \gamma^d b \mathcal{H}^d(E_{rec} \setminus E' \cap S) \\
&\quad + \frac{1+\varepsilon'}{1-\varepsilon'} J_F(E' \cap S) + (1+\varepsilon')^2 J_F(E_{irr} \cap S) \\
&\leq \left(1 + \frac{2\varepsilon'}{1-\varepsilon'} + 2\varepsilon' + \varepsilon'^2 + \frac{\gamma^d b \varepsilon'}{a}\right) J_F(E).
\end{aligned} \tag{3.1.8}$$

We put $E_1 = \psi_0(E)$, $D_i = H_i \cap B_i$. Since ψ_0 is indentity on S^c, we know that $\psi_0(E_{irr} \setminus S)$. Since $\psi_0(E \cap K_i) \subset D_i$, we know that $\psi_0(E \cap K_i)$ is rectifiable. Thus

$$(E_1)_{rec} \setminus \bigcup_{i=1}^{N} D_i \subset \psi_0(E_{rec} \setminus S) \cup \bigcup_{i=1}^{N} \psi_0(E \cap B_i \setminus K_i),$$

so

$$\begin{aligned}
\mathcal{H}^d\left((E_1)_{rec} \setminus \bigcup_{i=1}^{N} D_i\right) &\leq \mathcal{H}^d\left(\psi_0(E_{rec} \setminus S)\right) + \sum_{i=1}^{N} \mathcal{H}^d\left(\psi_0(E \cap B_i \setminus K_i)\right) \\
&\leq 2\varepsilon' \mathcal{H}^d(E).
\end{aligned}$$

Let $\alpha > 0$ be small number such that

$$\alpha < \min_{1 \leq i \leq N} \left(\frac{1}{2} r_i \rho_i\right).$$

For $1 \leq i \leq N$, we take a dyadic complex $\mathcal{S}_i$ of stride α in an orthonormal basis centered at x_i with d vectors parallel to H_i such that

$$K_i \subset |\mathcal{S}_i| \subset B\left(x_i, \left(1 + \frac{1}{2}\rho_i\right) r_i\right),$$

then for any $1 \leq i < j \leq N$,

$$\operatorname{dist}\left(|\mathcal{S}_i|, |\mathcal{S}_j|\right) \geq \min_{1 \leq k \leq N} \left(\frac{1}{2} r_k \rho_k\right),$$

we can apply Théorèm 2.3 in [21] (or Theorem 1 in [22]), there is a complex $\mathcal{S}$ such that $|\mathcal{S}| \subset U$, and that $\mathcal{S}_i$, $1 \leq i \leq N$ are subcomplexes of $\mathcal{S}$, and that $\mathcal{R}(S) \geq c \min\{\mathcal{R}(\mathcal{S}_1), \cdots, \mathcal{R}(\mathcal{S}_N)\}$, where $c > 0$ is a constant only depends on n, and we denote by $\mathcal{R}(\mathcal{S})$ the rotondity of complex S which is defined in [22, p.8] or [21, Définition 1.2.25].

Now, we do a Federer-Fleming projection which maps E_1 into d-skeleton as in [22, p.33], we can get a Lipschitz map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi|_{U^c} = \mathrm{id}_{U^c}$ such that $\psi(E_1)$ is contained in the union of d-skeleton of $\mathcal{S}$, and $\psi((E_1)_{irr})$ is still d-irregular, and

$$\psi\left(\bigcup_{i=1}^N D_i\right) \subset \bigcup_{i=1}^N D_i,$$

this is because that $\cup_{i=1}^N D_i$ is contained in the union of d-skeleton of $\mathcal{S}$, thus

$$\mathcal{H}^d(\psi((E_1)_{irr})) = 0$$

and there exist a constant C only depend on n and d such that

$$\mathcal{H}^d\left(\psi\left((E_1)_{rec} \setminus \bigcup_{i=1}^N D_i\right)\right) \leq C\mathcal{H}^d\left((E_1)_{rec} \setminus \bigcup_{i=1}^N D_i\right) \leq 2C\varepsilon'\mathcal{H}^d(E).$$

We take $\phi = \psi \circ \psi_0$, then

$$J_F(\psi(E_1)) \leq \left(1 + \frac{bC\varepsilon'}{a}\right) J_F(E_1),$$

thus

$$J_F(\phi(E)) \leq \left(1 + \frac{bC\varepsilon'}{a}\right)\left(1 + \frac{2\varepsilon'}{1-\varepsilon'} + 2\varepsilon' + \varepsilon'^2 + \frac{\gamma^d b\varepsilon'}{a}\right) J_F(E).$$

We let ε' tend to 0, we get that

$$J_F(\phi(E)) \leq (1+\varepsilon)J_F(E).$$

$\square$

Using this theorem, we can prove the following lemma.

Lemma 3.13. *Suppose that $0 < d < n$ and that $U \subset \mathbb{R}^n$. Suppose that F is an integrand. Then there is a positive constant $M' > 0$ depending only on d and n such that for all relatively closed set $E \subset U$ with locally finite Hausdorff measure, for all relatively compact subset $V \subset U$ and all $\epsilon > 0$, we can find a n-dimensional complex $\mathcal{S}$ and a subset $E'' \subset U$ satisfying the following properties:*

(1) *E'' is a* $\operatorname{diam}(U)$*-deformation of E over U, by putting $W = |\mathring{\mathcal{S}}|$ the interior of $|\mathcal{S}|$ we have $V \subset W \subset \overline{W} \subset U$, and there is a d-dimensional skeleton $\mathcal{S}'$ of $\mathcal{S}$ such that $E'' \cap \overline{W} = |\mathcal{S}'|$;*

(2) *$J_F(E'') \leq (1+\epsilon)J_F(E)$;*

(3) *there are $d+1$ complexes $\mathcal{S}^0, \ldots, \mathcal{S}^d$ such that $\mathcal{S}^\ell$ is contained in the ℓ-skeleton of $\mathcal{S}$ and there is a decomposition*

$$E'' \cap W = E^d \sqcup E^{d-1} \sqcup \ldots \sqcup E^0,$$

where for each $0 \leq \ell \leq d$,

$$E^\ell \in QM(W^\ell, M', \operatorname{diam}(W^\ell), \mathcal{H}^\ell),$$

where

$$\begin{cases} W^d = W \\ W^{\ell-1} = W^\ell \setminus E^\ell \end{cases} \qquad\qquad \begin{cases} E^d = \left|\mathcal{S}^d\right| \cap W^d \\ E^\ell = \left|\mathcal{S}^\ell\right| \cap W^\ell. \end{cases}$$

The proof of this lemma is also the same as the proof of the Lemme 5.2.6 in [21] or the Lemma 9 in [22], therefore we omit the proof. This lemma is very useful to seek a minimizer.

We now go to prove Lemma 3.2. In fact, Proposition 3.12 will be used in the proof. Let us recall the lemma.

Lemma 3.2. *If $d = 1$, then any almost polyhedral convex integrand $F : \mathbb{R}^n \times G(n,d) \to \mathbb{R}$ is a generalized elliptic integrand.*

Proof. Let ℓ be a line, and $x \in \ell$ be any point. For any $r > 0$, we denote by X_r and Y_r the two endpoints of the segment $\ell \cap \overline{B(x,r)}$. Let $S \supset \{X_r, Y_r\}$ be any compact set such that $S \subset \overline{B(x,r)}$ and S cannot be mapped into $\{X_r, Y_r\}$ by any Lipschitz map which leaves X_r and Y_r fixed.

For any $\varepsilon > 0$ small enough, by Proposition 3.12, we can find a polyheral complex $\mathcal{K}$ and a Lipschitz map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi = \operatorname{id}$ out of $B(0x, r+\varepsilon)$, $\|\phi - \operatorname{id}\| \leq \varepsilon r$, $\phi(S)$ is contained in the union of 1-dimensional skeletons of $\mathcal{K}$, and that

$$J_F(\phi(S)) \leq (1+\varepsilon r)J_F(S).$$

Since S cannot be mapped into $\{X_r, Y_r\}$ by any Lipschitz map which leaves X_r and Y_r fixed, we get that $\phi(S)$ also cannot be mapped into $\{\phi(X_r), \phi(Y_r)\}$ by any Lipschitz map which leaves $\phi(X_r)$ and $\phi(Y_r)$ fixed. Since $\phi(S)$ consists of segment, we can get that that there is a path γ from $\phi(X_r)$ to $\phi(Y_r)$, which consists of segments, suppose that is $\gamma = A_1 A_2 \cdots A_m$, $A_1 = \phi(X_r)$, $A_m = \phi(Y_r)$. By the definition of the almost polyhedral convex, we get that

$$J_F(A_1A_m) \leq J_F(\gamma) + h(r+\varepsilon)(r+\varepsilon) \leq (1+\varepsilon)J_F(S) + h(r+\varepsilon)(r+\varepsilon).$$

Let π' be the line which throug A_1 and A_m. Since $\|\phi - \mathrm{id}\| \leq \varepsilon r$, we get that $d_G(\pi, \pi') \leq 2\varepsilon$. We let $\varepsilon \to 0$, we will get that $J_F(A_1A_m) \to J_F(X_rY_r)$ and that

$$J_F(X_rY_r) \leq J_F(S) + h(r)r.$$

Thus F is a generalized elliptic integrand. $\square$

3.1.3 Lower semicontinuity of integral of an integrand on almost minimal sequence

Let W_1 and W_2 be two subspace of $\mathbb{R}^n$, $L : W_1 \to W_2$ be a linear map. $d = \dim(W_1)$. The d-dimensional jacobian $J_d(L)$ of L is defined by

$$J_d(L) = \| \wedge_d L\|.$$

In fact, the number $\mathcal{H}^d(L(A))/\mathcal{H}^d(A)$ does not depend on the choice of A when $\mathcal{H}^d(A) > 0$, which is exact the jacobian of L.

Let $E \subset \mathbb{R}^n$ be a d-dimensional rectifiable set. Suppose that V is a d-plane in $\mathbb{R}^n$, i.e. a translation of a d-dimensional subspace. We let $P_V : \mathbb{R}^n \to V$ be the orthogonal projection.

Lemma 3.14. *Let E, V and P_V be as above, let $\varphi = P_V|_E : E \to V$. If the tangent plane T_xE of E at point $x \in E$ exists, then*

$$ap J_d\varphi(x) = J_d(DP_V(x)|_{T_xE}) \tag{3.1.9}$$

and

$$d_G(T_xE, V) \leq \sqrt{2} \cdot \sqrt{1 - apJ_d\varphi(x)}, \tag{3.1.10}$$

where we denote by $apJ_d\varphi(x)$ the approximate jacobian of φ at point x, see [19, Chapter 3].

Proof. Equation (3.1.9) comes directly from the definition of approxiamte tangent plane, see [19]. Let's prove inequality (3.1.10). Let V and W be any two d-dimensional subspace. We will show that

$$d_G(W, V)^2 + J_d(P_V|_W)^2 \leq 1.$$

Indeed, for any small $\varepsilon > 0$, we can find $w \in W$ with $|w| = 1$ such that

$$d_G(W, V) - \varepsilon \leq \mathrm{dist}(w, V) \leq d_G(W, V).$$

We take an orthogonal basis $w_1, w_2, \cdots, w_d$ of W with $w_1 = w$. Then

$$J_d(P_V|_W) = |P_V(w_1) \wedge \cdots \wedge P_V(w_d)| \le |P_V(w_1)| \cdots |P_V(w_d)| \le |P_V(w_1)|.$$

Thus

$$J_d(P_V|_W)^2 \le |P_V(w)|^2 = 1 - |w - P_V(w)|^2 \le 1 - (d_G(W,V) - \varepsilon)^2.$$

We let ε tend to 0, then we get that

$$J_d(P_V|_W)^2 \le 1 - d_G(W,V)^2.$$

Since P_V is a linear map, we have that $DP_V(x) = P_V$. Thus we get that

$$d_G(T_xE, V) \le \sqrt{1 - apJ_d\varphi(x)^2} \le \sqrt{2} \cdot \sqrt{1 - apJ_d\varphi(x)}.$$

□

Theorem 3.15. *Let F be an integrand. Let U be an open set. Let $\{E_k\}$ be a sequence of sets such that*

$$E_k \in GQM(U, M_k, \varepsilon_k, \delta).$$

Suppose that $M_k \to 1$, $\varepsilon_k \to 0$ and $E_k \to E$. Then

$$J_F(E) \le \liminf_{k\to\infty} J_F(E_k).$$

Proof. For simplify the proof, we assume that $M_k \ge M_{k+1}$, $\varepsilon_k \ge \varepsilon_{k+1}$ for all $k \ge 1$. Let $\varepsilon \in (0,1)$ be small enough. We take $\varepsilon' > 0$ and $\rho > 0$ such that $1 - (1-\rho)^d < \varepsilon/2$ and $\varepsilon' < \min\{\varepsilon/10, \varepsilon^2, \rho\}$.

By Lemma 3.3 in [8], we get that

$$E \in GQM(U, M_k, \varepsilon_k, \delta),\ \forall k \ge 1, \tag{3.1.11}$$

and for any ball $B(x,r) \subset U$,

$$\mathcal{H}^d(E \cap B(x,r)) \le \liminf_{k\to\infty} \mathcal{H}^d(E_k \cap B(x,r)). \tag{3.1.12}$$

From (3.1.11), we can see that E is almost minimal.

We always assume that

$$\liminf_{k\to\infty} \mathcal{H}^d(E_k) < \infty;$$

otherwise we have nothing to prove. Then

$$\mathcal{H}^d(E) \le \liminf_{k\to\infty} \mathcal{H}^d(E_k) < \infty.$$

Similar as in [8], we can see that E is rectifiable, thus for $\mathcal{H}^d$-almost every $x \in E$, $\theta_E(x) = 1$ and E has an approximate tangent plane T_xE, see for example [29]. Since E is Ahlfors-regular, see [8], by Lemma 3.7, T_xE is a true tangent plane. Thus we can find $E' \subset E$ with $\mathcal{H}^d(E \setminus E') = 0$ such that for any $x \in E'$, and any $\varepsilon' > 0$, there exists $r_1 = r_1(x, \varepsilon') > 0$ such that

$$(1-\varepsilon')\omega_d r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq (1+\varepsilon')\omega_d r^d \tag{3.1.13}$$

and

$$E \cap B(x,r) \subset \mathcal{C}(x, r, T_xE, \varepsilon'), \tag{3.1.14}$$

for any $r \in (0, r_1]$, where $\mathcal{C}(x,r,P,\varepsilon) = \{x \in B(x,r) \mid \operatorname{dist}(x,P) \leq \varepsilon\,|x|\}$.

We put $r_\rho = (1-\rho)r$, then

$$\mathcal{H}^d(E \cap B(x,r) \setminus B(x,r_\rho)) \leq (1+\varepsilon')\omega_d r^d - (1-\varepsilon')\omega_d r_\rho^d \leq \varepsilon \mathcal{H}^d(E \cap B(x,r)).$$

We consider the Lipschitz function $\psi_{\rho,r} : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_{\rho,r}(t) = \begin{cases} 1, & t \leq r_\rho \\ -\frac{1}{\rho r}(t-r), & r_\rho < t \leq r \\ 0, & t > r \end{cases}$$

and Lipschitz map $\Phi_{x,\rho,r} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi_{x,\rho,r}(y) = \psi_{\rho,r}(|y-x|)\Pi_x(y) + (1 - \psi_{\rho,r}(|y-x|))y,$$

where we denote by $\Pi_x : \mathbb{R}^n \to T_xE$ the orthobonal projection. It is easy to check that

$$\Phi_{x,\rho,r}|_{B(x,r_\rho)} = \Pi_x|_{B(x,r_\rho)}$$

and

$$\Phi_{x,\rho,r}|_{B(x,r)^c} = \mathrm{id}_{B(x,r)^c}.$$

We take $h \in (\varepsilon', \rho)$ and

$$A_{x,r,h} = \left\{ y \in \overline{B(x,r)} \;\middle|\; \operatorname{dist}(y, T_xE) \leq hr \right\}.$$

Then similar to Theorem 3.8, we can show that

$$\operatorname{Lip}\left(\Phi_{x,\rho,r}|_{A_{x,\rho,h}}\right) \leq 2 + \frac{h}{\rho}.$$

Since E_k converge to E, and $E \cap B(x,r) \subset \mathcal{C}(x,r,T_xE,\varepsilon')$ for $r \in (0, r_1]$, and $h > \varepsilon'$, we get that $E_k \cap \overline{B(x,r)} \subset A_{x,r,h}$ when $k \geq k(x)$ and $0 < r < r_1$.

Since $E_k \in GQM(U, M_k, \varepsilon_k, \delta)$, and $M_k \to 1$, $\varepsilon_k \to 0$, we can see that

$$\Phi_{x,\rho,r}(E_k) \supset B(x, r_\rho) \cap T_x E,$$

when $0 < r < r_0 = \min\{r_1, \delta\}$ and k large enough (still assume that we have it when $k \geq k(x)$).

By Theorem 3.2.22 in [19], we have that

$$\int_{E_k \cap B(x,r_\rho)} apJ_d\Phi_{x,\rho,r}(y) d\mathcal{H}^d(y) = \int_V \# \left(\Phi_{x,\rho,r}^{-1}(z) \cap B(x, r_\rho)\right) d\mathcal{H}^d(z) \geq \omega_d r_\rho^d,$$

where we denote by V the tangent plane $T_x E$. By Lemma 3.14, we have that

$$\int_{E_k \cap B(x,r_\rho)} d_G(T_y E_k, V) d\mathcal{H}^d(y) \leq \sqrt{2} \int_{E_k \cap B(x,r_\rho)} \sqrt{1 - apJ_d\Phi_{x,\rho,r}(y)} d\mathcal{H}^d(y).$$

By low semicontinuous property, see for example Lemma 3.3 in [8], we get that

$$\mathcal{H}^d(E \cap B(x, r)) \leq \liminf_{k \to \infty} \mathcal{H}^d(E_k \cap B(x, r)). \tag{3.1.15}$$

Since $M_{k+1} \leq M_k$ and $\varepsilon_{k+1} \leq \varepsilon_k$, we get that

$$E_k \in GQM(\Omega, M_{k_0}, \varepsilon_{k_0}, \delta, \mathcal{H}^d), \text{ for } k \geq k_0.$$

By Lemma 3.12 in [8], we can get that

$$(1 + C\varepsilon_{k_0}) M_{k_0} \mathcal{H}^d(E \cap \overline{B(x, r)}) \geq \limsup_{k \to \infty} \mathcal{H}^d(E_k \cap \overline{B(x, r)}), \tag{3.1.16}$$

but we know that $M_k \to 1$,$\varepsilon_k \to 0$, we get that

$$\mathcal{H}^d(E \cap \overline{B(x, r)}) \geq \limsup_{k \to \infty} \mathcal{H}^d(E_k \cap \overline{B(x, r)}). \tag{3.1.17}$$

Thus, if $\mathcal{H}^d(E \cap \partial B(x, r)) = 0$, then

$$\mathcal{H}^d(E \cap B(x, r)) = \lim_{k \to \infty} \mathcal{H}^d(E_k \cap B(x, r)). \tag{3.1.18}$$

In fact, for $\mathcal{H}^1$-almost every $r \in (0, r_0)$, $\mathcal{H}^d(E \cap \partial B(x, r)) = 0$. For any $r \in (0, r_0)$, we can always arrange ρ such that $\mathcal{H}^d(E \cap \partial B(x, r_\rho)) = 0$.

Since (3.1.13) and (3.1.18), we can assume that when $k \geq k(x)$,

$$\mathcal{H}^d(E_k \cap B(x, r_\rho)) \leq (1 + \varepsilon')\mathcal{H}^d(E \cap B(x, r_\rho)) \leq (1 + 3\varepsilon')\omega_d r_\rho^d.$$

We put

$$I_{x,k} = \int_{E_k \cap B(x,r_\rho)} d_G(T_y E_k, V) d\mathcal{H}^d(y).$$

Then

$$
\begin{aligned}
I_{x,k} &\le \sqrt{2}\int_{E_k\cap B(x,r_\rho)} \sqrt{1-apJ_d\Phi_{x,\rho,r}(y)}d\mathcal{H}^d(y)\\
&\le \sqrt{2}(\omega_d r_\rho^d)^{\frac{1}{2}}\left(\int_{E_k\cap B(x,r_\rho)} 1-apJ_d\Phi_{x,\rho,r}(y)d\mathcal{H}^d(y)\right)^{\frac{1}{2}}\\
&\le 2\sqrt{\varepsilon'}\omega_d r_\rho^d.
\end{aligned}
\tag{3.1.19}
$$

We first consider that F is an Lipschitz integrand. That is, an integrand which satisfies that there exists a constant C such that for any $x_i, x_2 \in \mathbb{R}^n$ and any $\pi_1, \pi_2 \in G(n,d)$,

$$
|F(x_1,\pi_1)-F(x_2,\pi_2)| \le C(|x_1-x_2|+d_G(\pi_1,\pi_2)).
$$

We put

$$
\widetilde{I}_{x,r,\rho,k} = |J_F(E_k\cap B(x,r_\rho)) - F(x,V)\mathcal{H}^d(E_k\cap B(x,r_\rho))|
$$

Then

$$
\begin{aligned}
\widetilde{I}_{x,r,\rho,k} &= \left|\int_{E_k\cap B(x,r_\rho)} F(y,T_yE_k)-F(x,V)d\mathcal{H}^d(y)\right|\\
&\le \mathrm{Lip}(F)\int_{E_k\cap B(x,r_\rho)} |y-x|+d_G(T_yE_k,V)d\mathcal{H}^d(y)\\
&\le 2\mathrm{Lip}(F)(r_\rho+\sqrt{\varepsilon'})\omega_d r_\rho^d.
\end{aligned}
$$

Similarly, we can get that

$$
|J_F(E\cap B(x,r_\rho)) - F(x,V)\mathcal{H}^d(E\cap B(x,r_\rho))| \le 2\mathrm{Lip}(F)(r_\rho+\sqrt{\varepsilon'})\omega_d r_\rho^d.
$$

So we get that when $0<r<\min\{r_1(x,\varepsilon'),\delta,\varepsilon\}$ and $k\ge k(x)$,

$$
|J_F(E\cap B(x,r_\rho))-J_F(E_k\cap B(x,r_\rho))| \le 4\mathrm{Lip}(F)\varepsilon\omega_d r_\rho^d \le 8\mathrm{Lip}(F)\varepsilon\mathcal{H}^d(E\cap B(x,r_\rho)).
$$

We consider a family a balls

$$
\mathcal{B} = \{B(x,r_\rho) \mid x\in E'',\ \text{and } r,\rho \text{ are as above}\}.
$$

It is a Vitali covering of E'. By a Vitali covering theorem, see for example [???], we can find a finite or countably infinite disjoint balls

$$
\{B_i\}_{i\in I}\subset\mathcal{B}
$$

such that

$$\mathcal{H}^d\left(E' \setminus \bigcup_{i\in I} B_i\right) = 0.$$

We can find a finitely many balls $\{B_i\}_{i\in J} \subset \{B_i\}_{i\in I}$ such that

$$\mathcal{H}^d\left(E' \setminus \bigcup_{i\in J} B_i\right) \leq \varepsilon \mathcal{H}^d(E').$$

We denote $B_i = B(x_i, r_i)$ when $i \in J$. When $k \geq \max\{k(x_i) : i \in J\}$, we get that

$$\begin{aligned}
J_F(E) = J_F(E') &= J_F\left(E' \setminus \bigcup_{i\in J} B_i\right) + J_F\left(E' \cap \bigcup_{i\in J} B_i\right) \\
&\leq \sum_{i\in J} J_F\left(E \cap B_i\right) + \varepsilon(\sup F)\mathcal{H}^d(E) \\
&\leq \sum_{i\in J}(E_k \cap B_i) + \varepsilon(\sup F)\mathcal{H}^d(E) + \sum_{i\in J} 8\mathrm{Lip}(F)\varepsilon\mathcal{H}^d(E\cap B_i) \\
&\leq \sum_{i\in J}(E_k \cap B_i) + (\sup F + 8\mathrm{Lip}(F))\mathcal{H}^d(E)\varepsilon \\
&\leq J_F(E_k) + (\sup F + 8\mathrm{Lip}(F))\mathcal{H}^d(E)\varepsilon.
\end{aligned}$$

we let ε tend to 0, we get that

$$J_F(E) \leq \liminf_{k\to\infty} J_F(E_k).$$

We now suppose that F is any integrand. Let $\{U_m\}$ be a sequence of open set such that $U_m \subset U_{m+1}$ for any $m \geq 1$, $\bigcup U_m = U$, and each U_m is bounded.

Since U_m is bounded, we can find a sequence of Lipschitz integrands $\{F_\ell\}$ such that F_ℓ converges uniformly to F on U_m. Since F_ℓ is Lipschitz, we can see that

$$J_{F_\ell}(E \cap U_m) \leq \liminf_{k\to\infty} J_{F_\ell}(E_k \cap U_m).$$

Thus we get that

$$J_F(E \cap U_m) \leq \liminf_{k\to\infty} J_F(E_k \cap U_m).$$

Hence

$$J_F(E) \leq \liminf_{k\to\infty} J_F(E_k).$$

□

In above theorem, the integrand F can be any continuous integrand, but we require some condition on the sequence of sets $\{E\}_k$. In fact, for general integrand we could not expect an lower semicontiuous property without any assumption on the sequence of sets.

Example 3.16. For simplicity, we consider $n = 2$, $d = 1$, and take

$$U = \mathbb{R}^2 \setminus \{(0,0),(1,0)\}.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a positive continuous function such that $g(\theta) = g(\pi - \theta) = g(\pi + \theta)$ for any $\theta \in \mathbb{R}$ and that $\sqrt{2}g(\frac{\pi}{4}) < g(0)$. We consider the integrand F defined by

$$F(x,\theta) = g(\theta).$$

It quite easy to see that F is not generalized elliptic integrand.

We put

$$A_{k,i} = \left(\frac{2i}{2^k}, 0\right),\ B_{k,i} = \left(\frac{2i+1}{2^k}, \frac{1}{2^k}\right).$$

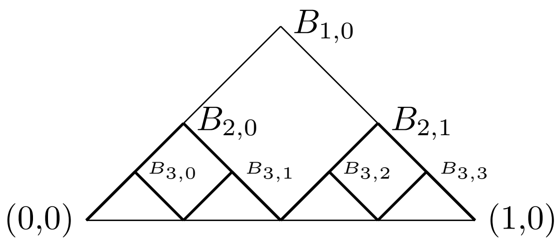


Figure 3.3: quasiminimal sets E_k

Let E_k be the union of segments $[A_i, B_i]$ and $[B_i, A_{i+1}]$, $0 \le i \le 2^{k-1}$. That is,

$$E_k = [A_0, B_0] \cup [B_0, A_1] \cup [A_1, B_1] \cup \cdots \cup [B_{2^{k-1}-1}, A_{2^{k-1}}].$$

We can easily check that

$$E_k \in QM(U, \sqrt{2}, \infty, \mathcal{H}^1),$$

and that E_k converges to the segment E which join the points $(0,0)$ and $(0,1)$ in Hausdorff distance.

We can see that

$$J_F(E) = g(0),$$

and that

$$J_F(E_k) = \sqrt{2}g\left(\frac{\pi}{4}\right).$$

Thus

$$J_F(E) > \liminf_{k\to\infty} J_F(E_k).$$

$\square$

3.2 Existence of minimizers under Reifenberg homological conditions

In this subsection, we will see some existence results. Let $B \subset \mathbb{R}^n$ be a compact set. Let $\mathscr{C}$ be a class of compact subsets in $\mathbb{R}^n$. Let F be an integrand. We set

$$m(\mathscr{C}, F) = \inf\{J_F(E \setminus B) \mid E \in \mathscr{C}\}.$$

Theorem 3.17. *Let F be a generalized elliptic integrand. If $\mathscr{C}$ a class of compact subsets in $\mathbb{R}^n$ which satisfies the following conditions:*

(1) *For any deformation $\{\varphi_t\}_{0\le t\le 1}$ in $\mathbb{R}^n \setminus B$ and any $E \in \mathscr{C}$, we have that $\varphi_1(E) \in \mathscr{C}$;*

(2) *For any sequence $\{E_k\}_{k=1}^{\infty} \subset \mathscr{C}$ such that E_k converges to some compact set E in Hausdorff distance, we have that $E \in \mathscr{C}$.*

Then we can find $E \in \mathscr{C}$ such that $J_F(E \setminus B) = m(\mathscr{C}, F)$.

Of course the problem will only be interesting when $m(\mathscr{C}, F) < +\infty$, which is usually fairly easy to arrange. We subtracted B because this way we shall not need to assume that $\mathcal{H}^d(B) < +\infty$, but of course if $\mathcal{H}^d(B) < +\infty$ we could replace $J_F(S \setminus B)$ with $J_F(S)$ in the definition.

Proof. We claim that we can find a ball $B(0, R)$ and a sequence of compact sets $(E_k)_{k\ge 1} \subset \mathscr{C}$ such that $B \subset B(0, R)$, $E_k \subset B(0, R)$ and

$$J_F(E_k \setminus B) \to m(\mathscr{C}, F).$$

We take any sequence of compact sets $(E'_k)_{k\ge 1}$ in $\mathscr{C}$ such that

$$J_F(E'_k \setminus B) \to m(\mathscr{C}, F).$$

We take

$$U'_k = \{x \in B(0, R_k) \mid \operatorname{dist}(x, B) > 2^{-k}\},$$

where

$$R_k > \max\{k, R_{k-1} + 1, \operatorname{dist}(0, E'_k) + \operatorname{diam}(E'_k) + 1\}.$$

By lemma 3.13, we can find a Lipschitz map $\phi_k' : \mathbb{R}^n \to \mathbb{R}^n$ and a complex $\mathcal{S}_k$ such that

$$\phi_k'|_{U_k'^c} = \mathrm{id}_{U_k'^c},\ U_{k-1}' \subset |\mathcal{S}_k| \subset U_k',$$

and

$$\phi_k'(E_k') \cap W_k' = F_k \sqcup F_k',$$

where $W_k' = |\mathring{\mathcal{S}}_k|$ and

$$F_k \in QM(W_k', M, \mathrm{diam}(W_k'), \mathcal{H}^d),$$

and F_k' is contained in the union of $(d-1)$-dimensional skeleton of $\mathcal{S}_k$.

We now prove that $(F_k)_{k\geq 1}$ is bounded, i.e. we can find a large ball $B(0,r)$ such that $B \cup (\cup_k F_k) \subset B(0,r)$. Suppose not, that is, suppose that for any large number $r > R_1$ there exist $k > 4r$ such that $F_k \setminus B(0,2r) \neq \emptyset$. If $x \in F_k \setminus B(0,2r)$, we take a cube Q centered at x with $\mathrm{diam}(Q) = r$, then by using Proposition 4.1 in [14], we have that

$$\mathcal{H}^d(F_k \cap Q) \geq C^{-1}\mathrm{diam}(Q)^d,$$

where C only depend on n and M. If we take r large enough, for example

$$r^d > \frac{2C}{\inf F}(m(\mathscr{C}, F) + 1),$$

and take k large enough such that $J_F(E_k') < m(\mathscr{C}, F) + 1$, then

$$\begin{aligned} C^{-1}r^d &\leq \mathcal{H}^d(F_k \cap Q) \\ &\leq \frac{1}{\inf F} J_F(\phi_k'(E_k')) \\ &\leq \frac{(1+2^{-k})}{\inf F} J_F(E_k') \\ &< \frac{2}{\inf F}(m(\mathscr{C}, F) + 1), \end{aligned}$$

this is a contradiction. Thus $\cup_k F_k$ is bounded. It is easy to see that $\cup_k(\phi_k'(E_k') \cap W_k'^c)$ is bounded, so we can assume that both $B \cup (\cup_k F_k)$ and $\cup_k(\phi_k'(E_k') \cap W_k'^c)$ are contained in a large ball $B(0,R)$. We take map $\rho : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\rho(x) = \begin{cases} x, & x \in B(0,R) \\ \frac{R}{|x|}x, & x \in B(0,R)^c, \end{cases}$$

ρ is 1-Lipschitz map. We put $E_k = \rho \circ \phi_k'(E_k')$, then $E_k \in \mathscr{C}$, and

$$E_k = (\phi_k'(E_k') \cap W_k'^c) \cup F_k \cup \rho(F_k').$$

Since $\mathcal{H}^d(F_k') = 0$, we have that

$$J_F(E_k \setminus B) = J_F(\phi_k'(E_k') \setminus B) \leq (1 + 2^{-k})J_F(E_k' \setminus B),$$

therefore

$$J_F(E_k \setminus B) \to m(\mathscr{C}, F),$$

and $(E_k)_{k\geq 1}$ is a sequence which we desire.

If $J_F(E_k \setminus B) = 0$ for some $k \geq 1$, then $m(\mathscr{C}, F) = 0$ and E_k is a minimizer, we have nothing to prove. We now suppose that for all $k \geq 1$, $0 < J_F(E_k \setminus B) < +\infty$. Thus $0 < \mathcal{H}^d(E_k \setminus B) < +\infty$.

We put

$$U = B(0, R+1) \setminus B,\ V_k = \{x \in B(0, R+1-2^{-k}) \mid \operatorname{dist}(x, B) > 2^{-k}\}.$$

By lemma 3.13, we can find polyhedral complexes $\mathcal{S}_k$, Lipschitz maps $\phi_k : \mathbb{R}^n \to \mathbb{R}^n$ and a constant $M' = M'(n, d)$ such that

(1) $V_k \subset |\mathcal{S}_k| \subset V_{k+1}$, $\phi_k|_{V_{k+1}^c} = \mathrm{id}_{V_{k+1}^c}$, and there exists a d-dimensional skeleton $\mathcal{S}_k'$ of $\mathcal{S}_k$ such that $E_k'' \cap W_k = |\mathcal{S}_k'|$, where $E_k'' = \phi_k(E_k)$ and $W_k = |\mathcal{S}_k|$;

(2) $J_F(E_k'' \setminus B) \leq (1 + 2^{-k})J_F(E_k \setminus B)$;

(3) there exist complexes $\mathcal{S}_k^0, \dots, \mathcal{S}_k^d$ such that $\mathcal{S}_k^\ell$ is contained in the ℓ-skeleton of $\mathcal{S}_k$ and there is a disjoint decomposition

$$E_k'' \cap \mathring{W}_k = E_k^d \sqcup E_k^{d-1} \sqcup \cdots \sqcup E_k^0,$$

where for each $0 \leq \ell \leq d$,

$$E_k^\ell \in QM(W_k^\ell, M', \operatorname{diam}(W_k^\ell), \mathcal{H}^\ell),$$

where

$$\begin{cases} W_k^d = \mathring{W}_k \\ W_k^{\ell-1} = W_k^\ell \setminus E_k^\ell \end{cases} \qquad \begin{cases} E_k^d = \left|\mathcal{S}_k^d\right| \cap W_k^d \\ E_k^\ell = \left|\mathcal{S}_k^\ell\right| \cap W_k^\ell, \end{cases}$$

and $\mathring{W}_k$ is the interior of W_k.

We note that for each k, E_k and W_k are two compact subsets of $\mathbb{R}^n$, thus $E_k'' \cap W_k$ is a compact subset of $\mathbb{R}^n$. We may suppose that $E_k'' \cap W_k \to E'$ in Hausdorff distance, passing to a subsequence if necessary. We put $E = E' \cup B$. We will show that E is a minimizer.

First of all, we show that E is in the class $\mathscr{C}$, i.e. $E \in \mathscr{C}$. Since ϕ_k is Lipschitz map and $\phi_k|_{V_{k+1}^c} = \mathrm{id}_{V_{k+1}^c}$, in particular, $\phi_k|_B = \mathrm{id}_B$, thus $E_k'' = \phi_k(E_k) \in \mathscr{C}$. Since $V_k \subset W_k \subset V_{k+1}$, we have that

$$\phi_k(E_k) \setminus W_k \supset \phi_k(E_k) \setminus V_{k+1} \supset B,$$

and

$$\phi_k(E_k) \setminus W_k \subset \phi_k(E_k) \setminus V_k \subset B(2^{-k}),$$

where we denote by $B(\epsilon)$ denote the ϵ-neighborhood of B. Thus $E_k'' \setminus W_k \to B$ in Hausdorff distance, so that

$$E_k'' = (E_k'' \cap W_k) \cup (E_k'' \setminus W_k) \to E' \cup B = E,$$

and we have that $E \in \mathscr{C}$.

Next, we will show that $J_F^d(E \setminus B) = m(\mathscr{C}, F)$.

Passing to a subsequence if necessary, we may assume that

$$E_k^d \to E^d \text{ in } U, \text{ for } 0 \leq \ell \leq d,$$

For any $0 \leq \ell \leq d$, we put

$$U^\ell = U \setminus \bigcup_{\ell < \ell' \leq d} E^{\ell'},$$

we assume that $E_k^\ell \to E^\ell$ in U. Then

$$E \setminus B = \bigcup_{0 \leq \ell \leq d} E^\ell.$$

Since

$$E_k^d \in QM(W_k^d, M', \mathrm{diam}(W_k^d), \mathcal{H}^d),$$

we can apply the Theorem 3.8, and get that

$$J_F(E^d \cap W_k^d) \leq \liminf_{m \to \infty} J_F(E_m^d \cap W_k^d) \leq \liminf_{m \to \infty} J_F(E_m^d).$$

Since $V_k \subset W_k \subset V_{k+1}$ and $W_k^d = \mathring{W}_k$, we have

$$\bigcup_k W_k^d = \bigcup_k V_k = U,$$

thus

$$J_F(E^d) \leq \liminf_{m \to \infty} J_F(E_m^d).$$

For any $0 \le \ell \le d$, for any $\varepsilon > 0$, we put $U^d_\varepsilon = B(0, R+1-\varepsilon) \cap U^d$ and

$$U^\ell_\varepsilon = \left\{ x \in B(0, R+1-\varepsilon) \,\middle|\, \operatorname{dist}\left(x, \bigcup_{\ell < \ell' \le d} E^{\ell'}\right) > \varepsilon \right\}.$$

Then $U^\ell_{\varepsilon_1} \subset U^\ell_{\varepsilon_2}$ for any $0 < \varepsilon_2 < \varepsilon_1$, and

$$\bigcup_{\varepsilon > 0} U^\ell_\varepsilon = U^\ell.$$

Since $E^\ell_k \to E^\ell$ in U, we have that $E^\ell_k \cap U^\ell_\varepsilon \to E^\ell \cap U^\ell_\varepsilon$ in U^ℓ_ε. We will show that for any $\varepsilon > 0$, there exists k_ε such that for $k \ge k_\varepsilon$,

$$E^\ell_k \cap U^\ell_\varepsilon \in QM(U^\ell_\varepsilon, M', \operatorname{diam}(U^\ell_\varepsilon), \mathcal{H}^\ell).$$

Indeed, for any $\varepsilon > 0$, we can find k_ε such that $U^\ell_\varepsilon \subset W^\ell_k$. We prove this fact by induction on ℓ.

First, we take a positive integer k_ε such that $2^{-k_\varepsilon} < \varepsilon$, then $U^d_\varepsilon \subset W^d_k$ for any $k \ge k_\varepsilon$.

Next, we suppose that there is an integer k_ε such that $U^\ell_\varepsilon \subset W^\ell_k$ for $k \ge k_\varepsilon$. Since $E^\ell_k \to E^\ell$ in U^ℓ and

$$W^{\ell-1}_k = W^\ell_k \setminus E^\ell_k,\ U^\ell_\varepsilon = \left\{ x \in U^\ell_\varepsilon \mid \operatorname{dist}\left(x, E^\ell\right) > \varepsilon \right\},$$

we can find k'_ε such that $U^{\ell-1}_\varepsilon \subset W^{\ell-1}_k$ for $k \ge k'_\varepsilon$.

Since $U^\ell_\varepsilon \subset W^\ell_k$ and

$$E^\ell_k \in QM(W^\ell_k, M', \operatorname{diam}(W^\ell_k), \mathcal{H}^\ell),$$

we get that

$$E^\ell_k \cap U^\ell_\varepsilon \in \mathbf{QM}(U^\ell_\varepsilon, M', \operatorname{diam}(U^\ell_\varepsilon), \mathcal{H}^\ell).$$

For any $\delta > 0$, we put $\Omega_\delta = \{x \in U_\varepsilon \mid \operatorname{dist}(x, U^c_\varepsilon) \ge 10\delta\}$. $E^\ell_k \cap \Omega_\delta$ is a compact set, and $\{B(x,\delta) \mid x \in E^\ell_k \cap \Omega_\delta\}$ is an open covering of $E^\ell_k \cap \Omega_\delta$, we can find a finitely many balls $\{B(x_i,\delta)\}_{i \in I}$ which is a covering of $E^\ell_k \cap \Omega_\delta$, by the 5-covering lemma, see for example the Theorem 2.1 in [29], we can find a subset $J \subset I$ such that $B(x_{j_1}, \delta) \cap B(x_{j_2}, \delta) = \emptyset$ for $j_1, j_2 \in J$ with $j_1 \ne j_2$, and

$$\bigcup_{i \in I} B(x_i, \delta) \subset \bigcup_{j \in J} B(x_j, 5\delta).$$

Since $B(x_{j_1}, \delta) \cap B(x_{j_2}, \delta) = \emptyset$ for $j_i, j_2 \in J$, we have that

$$\mathcal{L}^n(U_\varepsilon) \ge \sum_{j \in J} \mathcal{L}^n(B(x_j, \delta)),$$

thus

$$\#J \leq \frac{\mathcal{L}^n(U_\varepsilon)}{\omega_n \delta^n}.$$

By the Proposition 4.1 in [14], we have that

$$C^{-1}(5\delta)^\ell \leq \mathcal{H}^\ell(E_k^\ell \cap B(x_j, 5\delta)) \leq C(5\delta)^\ell,$$

so

$$\mathcal{H}^\ell(E_k^\ell \cap \Omega_\delta) \leq \sum_{j\in J} \mathcal{H}^\ell(E_k^\ell \cap B(x_j, 5\delta)) \leq \sum_{j\in J} C(5\delta)^\ell \leq \omega_n^{-1}\mathcal{L}^n(U_\varepsilon)5^d\delta^{\ell-n}C.$$

Applying the theorem 3.4 in [7], we get that

$$\mathcal{H}^\ell(E^\ell \cap \Omega_\delta) \leq \liminf_{k\to\infty} \mathcal{H}^\ell(E_k^\ell \cap \Omega_\delta) \leq \omega_n^{-1}\mathcal{L}^n(U_\varepsilon)5^d\delta^{\ell-n}C,$$

and $\dim_{\mathcal{H}} E^\ell \cap \Omega_\delta \leq \ell$, hence $\dim_{\mathcal{H}} E^\ell \leq \ell$, thus $\mathcal{H}^d(E^\ell) = 0$.

we get that

$$\begin{aligned}
J_F(E \setminus B) &= J_F(E^d) \\
&\leq \liminf_{k\to\infty} J_F(E_k^d) \\
&\leq \liminf_{k\to\infty} J_F(E_k'' \setminus B) \\
&\leq \liminf_{k\to\infty} (1+2^{-k}) J_F(E_k \setminus B) \\
&= \liminf_{k\to\infty} J_F(E_k \setminus B) \\
&= m(\mathscr{C}, F).
\end{aligned}$$

Since $E \in \mathscr{C}$, we have that

$$J_F(E \setminus B) \geq m(\mathscr{C}, F),$$

therefore

$$J_F(E \setminus B) = m(\mathscr{C}, F).$$

□

The following proposition is taken from [34, 3.1 Proposition].

Proposition 3.18. *Let $B \subset \mathbb{R}^n$ be a compact subset. Suppose that for $j = 1, 2, \ldots,$ $S_j \subset \mathbb{R}^n$ is a compact set with $B \subset S_j$, and that S_j converge in Hausdorff distance to a compact set $S \subset \mathbb{R}^n$. Let $L \subset \check{H}_{k-1}(B; G)$ be a subgroup such that $L \subset \ker \check{H}_{k-1}(i_{B,S_j})$. Then $L \subset \ker \check{H}_{k-1}(i_{B,S})$.*

The proof of the proposition is essentially the same as the proof of Proposition 3.1 in [34], so we omit the proof.

We now to prove our existence result for Reifenberg's Plateau problem. It can be deduced from Theorem 3.17 and Proposition 3.18.

Let $B \subset \mathbb{R}^n$ be a given compact set, G be an abelian group. If S is another compact set that contains B, we shall denote by $i_{B,S} : B \to S$ the natural inclusion, by $H_k(i_{B,S}) : H_k(B;G) \to H_k(S;G)$ the corresponding homomorphism between homology groups, and by $\check{H}_k(i_{B,S}) : \check{H}_k(B;G) \to \check{H}_k(S;G)$ the corresponding homomorphism between Čech homology groups. Let L be a subgroup of $\check{H}_{d-1}(B;G)$. Recall that a compact set $S \supset B$ is called with algbraic boundary containing L if $L \subset \ker \check{H}_{d-1}(i_{B,S})$.

A simple case is when L is the full group $\check{H}_k(B;G)$; then $S \supset B$ spans L in Čech homology precisely when the mapping $H_k(i_{B,S})$ is trivial. But it may be interesting to study other other subgroups L, and this will not make the proofs any harder.

We have a similar definition which we just replace $\check{H}_{d-1}(i_{B,S})$ with $H_{d-1}(i_{B,S})$. It would be very nice if our existence theorem was in terms of singular homology, but unfortunately we cannot prove the corresponding statement at this time.

We set

$$\mathscr{C}_{\text{Čech}}(B,G,L) = \left\{ S \subset \mathbb{R}^n \;\middle|\; \begin{array}{c} S \supset B \text{ is a compact set with} \\ \text{algebraic boundary containing } L \end{array} \right\}$$

Theorem 3.19. *Let the compact set $B \subset \mathbb{R}^n$, a generalized integrand F, an abelian group G, and a subgroup L of $\check{H}_{d-1}(B;G)$ be given. Then there exists a compact set $E \in \mathscr{C}_{\check{C}ech}(B,G,L)$ such that*

$$J_F(E \setminus B) = m\left(\mathscr{C}_{\check{C}ech}(B,G,L), F\right).$$

As was mentioned before, this theorem was proved by Reifenberg in [38], under the additional assumption that G be compact.

Proof. We only need to prove that $\mathscr{C}_{\text{Čech}}(B,G,L)$ saitisfies the conditions in Theorem 3.17.

For any Lipschitz map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi|_B = \mathrm{id}_B$, and any compact set $E \in \mathscr{C}_{\text{Čech}}(B,G,L)$, we have $i_{B,\varphi(E)} = \varphi \circ i_{B,E}$, thus

$$\check{H}_{d-1}(i_{B,\varphi(E)})(L) = \check{H}_{d-1}(\varphi) \circ \check{H}_{d-1}(i_{B,E})(L) = 0,$$

this implies that

$$L \in \ker\left(\check{H}_{d-1}(i_{B,\varphi(E)})\right),$$

so

$$\varphi(E) \in \mathscr{C}_{\text{Čech}}(B,G,L).$$

For any sequence

$$\{E_k\}_{k=1}^{\infty} \subset \mathscr{C}_{\text{Čech}}(B,G,L),$$

if E_k converge to E in Hausdorff distance for some compact set $E \subset \mathbb{R}^n$, then by Proposition 3.18, we get that

$$E \in \mathscr{C}_{\text{Čech}}(B,G,L).$$

The class $\mathscr{C}_{\text{Čech}}(B,G,L)$ saitisfies the two conditions in Theorem 3.17, thus we can find a compact set $E \in \mathscr{C}_{\text{Čech}}(B,G,L)$ such that

$$J_F(E \setminus B) = m(\mathscr{C}_{\text{Čech}}(B,G,L),F).$$

$\square$

Let us look at an another related example called free boundary Plateau problem, see for example in [33]. Given a compact set $B \subset \mathbb{R}^n$, an abelian group G, and a subgroup L of $\check{H}_{d-1}(B;G)$. We call compact set $X \subset \mathbb{R}^n$ is a surface with free boundary including L, if

$$L \subset \check{H}_{d-1}(i_{X\cap B,B})\left(\ker \check{H}_{d-1}(i_{X\cap B,X})\right).$$

We set

$$\mathscr{C}_{\text{free}}(B,G,L) = \left\{ E \subset \mathbb{R}^n \;\middle|\; \begin{array}{c} E \text{ is a compact set with} \\ \text{free boundary including } L \end{array} \right\}$$

Theorem 3.20. *Let B, G, L and F ba as in Theorem 3.19. Then we can find $E \in \mathscr{C}_{free}(B,G,L)$ such that*

$$J_F(E \setminus B) = m\left(\mathscr{C}_{free}(B,G,L)\right).$$

Proof. We will show that $\mathscr{C}_{\text{free}}(B,G,L)$ saitisfies the two conditions in Theorem 3.17. It is fairly easy to verify the frist condition, so we omit it.

Let $\{E_n\}_{n=1}^{\infty} \subset \mathscr{C}_{\text{free}}(B,G,L)$ be a sequence of compact sets such that $E_n \to E$ in Hausdorff distance for some compact sets $E \subset \mathbb{R}^n$. We put

$$X_n = \left(\bigcup_{k\geq n} E_k\right) \bigcup E.$$

By Lemma 3 in [33], we know that $X_n \in \mathscr{C}_{\text{free}}(B,G,L)$, then we apply Lamma 4 in [33], we get that $E = \bigcap\limits_{n=1}^{\infty} X_n \in \mathscr{C}_{\text{free}}(B,G,L)$. $\square$

Here we have show that there exists (at least) a minimizer to minimize quantity

$$J_F(E \setminus B).$$

It would be more interesting to minimize

$$J_{F_1}(E \setminus B) + J_{F_2}(E \cap B)$$

for free boundary problem, where F_1 and F_2 are two integrand. But unfortunately we cannot prove the corresponding existence result here, because we do not have any lower semicontinuous property for $J_{F_2}(E \cap B)$.

It should be also interesting to regularity of minimizers for the Reifenberg Plateau problem. In fact, if the genralized elliptic integrand F donot depends on the direction, i.e.

$$F(x, \theta) = f(x), \text{ for all } x \in \mathbb{R}^n, \theta \in G(n, d),$$

for some continuous function $f : \mathbb{R}^n \to \mathbb{R}$ with $0 < c \leq f \leq C < \infty$, then any minimizer in Theorem 3.19 is almost minimal in $\mathbb{R}^n \setminus B$. But when F is a general generalized elliptic integrand, we donot know whether or not such a minimizer is almost minimal. Of course, we can get, from the proof, that such a minimizer is quasiminimal.

Chapter 4

Regularity of sliding almost minimal sets at the boundary

Recall that a gauge function is a nondecreasing function $h : [0, \infty] \to [0, \infty]$ with $\lim_{t\to 0} h(t) = 0$. Let $\delta > 0$ and an open set $U \subset \mathbb{R}^n$ be given. A δ-deformation in U is a family of maps $\{\varphi_t\}_{0\le t\le 1}$ from U into itself such that

$$\varphi_1 \text{ is Lipschitz and } \varphi_0 = \mathrm{id}_U,$$

the function

$$[0, 1] \times U \to U, (t, x) \mapsto \varphi_t(x)$$

is continuous, $\widehat{W}$ is relatively compact in U and $\mathrm{diam}(\widehat{W}) < \delta$, where

$$\widehat{W} = \bigcup_{t\in[0,1]} \left(W_t \cup \varphi_t(W_t)\right),\ W_t = \{x \in U; \varphi_t(x) \neq x\}. \tag{4.0.1}$$

We say that a relatively closed d-dimensional set $E \subset U$ is (U, h)-almost-minimal if it satisfies

$$\mathcal{H}^d(E \cap W_1) \le \mathcal{H}^d(\varphi(E \cap W_1)) + h(\delta)\delta^d,$$

for any δ-deformation $\{\varphi_t\}_{0\le t\le 1}$. In [39], Jean Taylor proved that if U is an open set in $\mathbb{R}^3$, E is a reduced (U, h)-almost-minimal set and $h(r) \le cr^\alpha$, then for any $x \in U$, there is a small neighborhood of x contained in U and in this neighborhood, E is C^1 diffeomorphic to a 2-dimensional minimal cone, while 2-dimensional minimal cones are planes, cones of type $\mathbb{Y}$ and cones of type $\mathbb{T}$.

In this chapter, we concentrate on boundary regularity, and always consider the following sliding boundary conditions. Let $\Omega \subset \mathbb{R}^n$ be a closed domain in $\mathbb{R}^n$. Let L_1 be a closed sets (it will be consider as the sliding boundary).

Definition 4.1. Let U be an open set. For $\delta > 0$, we say that a one parameter family $\{\varphi_t\}_{0\le t\le 1}$ of maps from U into itself is a δ-sliding-deformation in U, if it satisfies the following properties: $\varphi_0 = \mathrm{id}_U$, φ_1 is Lipschitz, $(t, x) \mapsto \varphi_t(x)$ is continuous on $[0, 1] \times U$, $\varphi_t(x) \in L_1$ for any $x \in L_1$ and any $t \in [0, 1]$, $\widehat{W}$ is relatively compact in U and $\mathrm{diam}(\widehat{W}) < \delta$.

Let $E \subset \Omega$ be closed in Ω; we say that a closed subset $F \subset \Omega$ is a competitor of E in U, if $F = \varphi_1(E)$ for some sliding deformation $\{\varphi_t\}_{0\leq t\leq 1}$ in U.

Definition 4.2. Let $E \subset \Omega$ be closed in U. We say that E is (U,h)-sliding-almost-minimal, if for each ball B, $\mathcal{H}^d(E\cap B) < +\infty$, and for each $\delta > 0$ and each δ-sliding-deformation $\{\varphi_t\}_{0\leq t\leq 1}$, we have

$$\mathcal{H}^d(E\cap W_1) \leq \mathcal{H}^d(\varphi_1(E\cap W_1)) + h(\delta)\delta^d, \tag{4.0.2}$$

where $W_1 = \{x \in U; \varphi_1(x) \neq x\}$.

We say that E is an A_+-sliding-almost-minimal set in U if under the same circumstances,

$$\mathcal{H}^d(E\cap W_1) \leq (1 + h(\delta))\mathcal{H}^d(\varphi_1(E\cap W_1)).$$

In the definition of (U,h)-sliding-almost-minimal set, we can replace inequality (4.0.2) by the inequality

$$\mathcal{H}^d(E \setminus \varphi_1(E)) \leq \mathcal{H}^d(\varphi_1(E) \setminus E) + h(\delta)\delta^d,$$

at least if that L_1 is not too bed, see [11].

When Ω, L_1 and gauge function h are clear, and $U = \mathbb{R}^n$. For simplicity, we may say that an (U,h)-sliding-almost-minimal set E is sliding almost minimal (in Ω with sliding boundary L_1). It quite easy to see that for any (U,h)-sliding-almost-minimal set E, $E \setminus L_1$ is $(U \setminus L_1, h)$-almost-minimal.

We say that E is (sliding) minimal in U if it is (sliding) almost minimal with gauge function $h = 0$, that is,

$$\mathcal{H}^d(E\cap W_1) \leq \mathcal{H}^d(\varphi_1(E\cap W_1))$$

or

$$\mathcal{H}^d(E \setminus \varphi_1(E)) \leq \mathcal{H}^d(\varphi_1(E) \setminus E)$$

for any (sliding) deformation $\{\varphi_t\}_{0\leq t\leq 1}$ in U.

We say that a d-dimensional set E is reduced if $E = E^*$, where

$$E^* = \{x \in E \mid \mathcal{H}^d(E \cap B(x,r)) > 0 \text{ for every } r > 0\}.$$

We can prove that

$$\mathcal{H}^d(E \setminus E^*) = 0$$

and that E^* is also (sliding) almost minimal when E is (sliding) almost minimal, see for instance [8, 11]. In this paper, we always assume that a sliding almost minimal set is reduced.

Recall that for any set E, any point $x \in E$, we have defined a function $\theta_E(x, \cdot)$ by setting

$$\theta_E(x, r) = \frac{\mathcal{H}^d(E \cap B(x, r))}{\omega_d r^d}.$$

If the limits

$$\lim_{r \to 0} \theta_E(x, r)$$

exists, we will denote it by $\theta_E(x)$ (or $\theta(x)$), and call it the density of E at the points x. A property of almost monotonicity of density for sliding almost minimal set will be often used. That is, Proposition 5.27 in [8] and Theorem 28.7 in [11]. We now put them together, it can be stated rough as follows:

$$\theta_E(x, r)e^{\lambda A(r)} \text{ is a nondecreasing function of } r, \tag{4.0.3}$$

when r small, where E is a sliding almost minimal set with gauge function h, $A(r) = \int_0^r h(2t)\frac{dt}{t}$, and λ is a constant that depends only on d and n. Let's refer to [11, Section 5] and [8, Section 28] for more detailed statement. It will be will be used frequently. From the fact (4.0.3), we can get that for any sliding almost minimal set E, $\theta_E(x)$ exists for every $x \in E$.

A blow-up limit of a set E at $x \in E$ is any closed set in $\mathbb{R}^n$ that can be obtained as the limit of a sequence $\{r_k^{-1}(E - x)\}$ with $\lim_{k\to\infty} r_k = 0$.

A set E in $\mathbb{R}^n$ is called a cone centered at origin 0 if for any $x \in E$ and any $t \geq 0$, $tx \in E$. A cone centered at $x \in \mathbb{R}^n$ is the translation of a cone centered at origin 0 to x.

Suppose that E is a sliding almost minimal set, and $x \in E$. If x is not contained in the sliding boundary, then any blow-up limit of E at x is a minimal cone in $\mathbb{R}^n$, see [8]; if x is in the sliding boundary, then any blow-up limit of E at x is a sliding minimal cone, see [11]. We refer to [8] and [11] for the basic properties of blow-up limits.

4.1 One dimensional sliding minimal sets in a half plane

In this section we discuss one dimensional sliding minimal sets in a half plane. We discuss the one dimensional case, because it is very easy, and the list of one dimensional sliding cones will be used to classify the two dimensional sliding minimal cones in a half space. For simplicity, we assume that

$$\begin{aligned} \Omega &= \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}, \\ L_1 &= \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}. \end{aligned} \tag{4.1.1}$$

For any $t \in \mathbb{R}$, and any $\alpha \in (0, \frac{\pi}{2})$, we set

$$P_t = \{(t, y) \mid y \geq 0\}$$

and

$$V_{\alpha,t} = \{(x, y) : y = |x \tan \alpha + t|\}.$$

It is very easy to see that the set P_t and $P_t \cup L_1$ are sliding minimal. It is also not hard to see that $V_{\alpha,t}$ is minimal if and only if $0 < \alpha \leq \frac{\pi}{6}$.

Lemma 4.3. *Let Ω and L_1 be as in (4.1.1). Suppose that E is a minimal cone in Ω with sliding boundary L_1 centered at 0. Then E is one of L_1, P_0, $P_0 \cup L_1$ and $V_{\alpha,0}$ for some $\alpha \in (0, \frac{\pi}{6})$.*

Proof. Let $K = E \cap \partial B(0, 1)$. We note that K is a finite set because otherwise $\mathcal{H}^1(E \cap B(0, 1)) = \infty$. Write $K = \{a_1, \cdots, a_n\}$, and denote by l_i the ray form 0 through the point a_i. Suppose $a_i, a_j \in \Omega \setminus L_1$, $i \neq j$. Similarly to (10.3) in Lemma 10.2 in [8], we can get that

$$\mathrm{Angle}(l_i, l_j) \geq \frac{2\pi}{3}.$$

Therefore, there are at most four point in K.

Case 1, if there is only one point in K, i.e. $K = \{a_1\}$. It is easy to see that $a_1 \neq (1, 0)$ and $a_1 \neq (-1, 0)$. If $a_1 = (0, 1)$, it is very easy to see that E is minimal. If $a_1 \neq (0, 1)$, we put $a_1 = (x, y)$, then

$$E' = \{(x, ty) \mid 0 \leq t \leq 1\} \cup \{(tx, ty) \mid t \geq 1\}$$

is a competitor of E, and $\mathcal{H}^1(E') < \mathcal{H}^1(E)$. Then E could not be minimal. In this case, $E = P_0$ is a ray which is perpendicular to L_1.

Case 2, there are two points in K, i.e. $K = \{a_1, a_2\}$. If $a_1 = (-1, 0)$ and $a_2 = (1, 0)$, then $E = L_1$. If $a_1 = (-1, 0)$ and $a_2 \neq (1, 0)$, then

$$E'' = (E \setminus B(0, 1)) \cup [a_1, a_2]$$

is a competitor of E and $\mathcal{H}^1(E'') < \mathcal{H}^1(E)$. Then E could not be minimal. If $a_2 = (1, 0)$ and $a_1 \neq (-1, 0)$, for the same reason as before, E is not minimal. If $a_1, a_2 \notin \{(-1, 0), (1, 0)\}$, we put $a_1 = (\cos \alpha_1, \sin \alpha_1)$, $a_2 = (\cos \alpha_2, \sin \alpha_2)$ and $\widetilde{a}_2 = (\cos \alpha_2, -\sin \alpha_2)$ with $0 < \alpha_2 < \alpha_1 < \pi$, then

$$\mathcal{H}^1([a_1, 0] \cap [0, a_2]) \geq \mathcal{H}^1([a_1, \widetilde{a}_2]),$$

and with equality if and only if $\alpha_1 + \alpha_2 = \pi$. It means that when $a_2 \neq (-\cos \alpha_1, \sin \alpha_1)$, E could not be minimal. We now suppose that $a_2 = (-\cos \alpha_1, \sin \alpha_1)$. Then $E = V_{\alpha_1,0}$, and $0 < \alpha_1 \leq \frac{\pi}{6}$ because E is minimal.

Case 3, there are three point in K. Write $K = \{a_1, a_2, a_3\}$, $a_1 = (x_1, y_1)$, $a_2 = (x_2, y_2)$ and $a_3 = (x_3, y_3)$, $x_1 < x_2 < x_3$. If $x_1 \neq -1$ and $x_3 \neq 1$, then

$$\mathrm{Angle}(l_1, l_2) \geq \frac{2\pi}{3} \text{ and } \mathrm{Angle}(l_2, l_3) \geq \frac{2\pi}{3}; \tag{4.1.2}$$

that is impossible. If $x_1 = -1$ and $x_3 \neq 1$, then

$$\mathrm{Angle}(l_2, l_3) \geq \frac{2\pi}{3},$$

thus $-1 < x_2 < -\frac{\sqrt{3}}{2}$. We can see that

$$E''' = (E \setminus B(0,1)) \cup [0, a_1] \cup [(x_2, 0), a_2] \cup [0, a_3]$$

is a competitor of E, and $\mathcal{H}^1(E''') < \mathcal{H}^1(E)$, thus E could not be minimal. Similarly, we can see that we cannot have $x_1 \neq 1$ and $x_3 = -1$. We now suppose that $x_1 = -1$ and $x_3 = 1$, i.e. $a_1 = (-1, 0)$ and $a_3 = (1, 0)$. If $x_2 \neq 0$, then

$$E'''' = L_1 \cup \{(x_2, ty_2) \mid 0 \leq t \leq 1\} \cup \{(tx_2, ty_2) \mid t \geq 1\}$$

is a competitor of E, and $\mathcal{H}^1(E'''') < \mathcal{H}^1(E)$, E is not minimal, impossible! If $x_2 = 0$, then $a_2 = (0, 1)$. That is, $E = P_0 \cup L_1$.

Case 4, there are four point in K. If there are at least three point in $\partial B(0,1) \cap \Omega \setminus L_1$, similarly to (4.1.2), that is impossible. Thus there at most two point in $\partial B(0,1) \cap \Omega \setminus L_1$, so there are exactly two point in $\partial B(0,1) \cap \Omega \setminus L_1$ and exactly two point in $\partial B(0,1) \cap \Omega \cap L_1$. We put $K = \{a_1, a_2, a_3, a_4\}$, $a_1 = (-1, 0)$, $a_2 = (x_2, y_2)$, $a_3 = (x_3, y_3)$, $a_4 = (1, 0)$, $\widetilde{a}_2 = (x_2, 0)$ and $\widetilde{a}_3 = (x_3, 0)$. Then

$$\mathcal{H}^1([a_2, 0] \cap [0, a_3]) > \mathcal{H}^1([a_2, \widetilde{a}_2]) + \mathcal{H}^1([a_3, \widetilde{a}_3]),$$

which means that E could not be minimal. □

Proposition 4.4. *Let Ω and L_1 be as in (4.1.1). Suppose that E is a sliding minimal set in Ω with sliding boundary L_1, and $E \supset L_1$. Then either $E = L_1$ or $E = P_t \cup L_1$ for some $t \in \mathbb{R}$.*

Proof. For $r > 0$, we put $E_r = \frac{1}{r} E$, $A_r = \Omega \cap \overline{B(0,r)}$ and $S_r = \Omega \cap \partial B(0,r)$.

We claim that there exists a sequence $\{r_n\}$, such that $r_n \to \infty$ and there are at most three point in $E \cap S_{r_n}$.

Since E is sliding minimal, we have that $\theta_E(0, r)$ is nondecreasing and bounded, see [11, Theorem 28.4]. Thus for any $\varepsilon > 0$, we can find $r_\varepsilon > 0$ such that $\theta_E(0, r) \geq \theta_E(0, \infty) - \varepsilon$ for $r \geq r_\varepsilon$, where we denote $\theta_E(0, \infty) =$

$\lim\limits_{r\to\infty} \theta_E(0,r)$. We can easily see that $\theta_{E_r}(0,t) = \theta_E(0,rt)$. If we take $r > 2r_\varepsilon$, then

$$\theta_{E_r}(0,t) = \theta_E(0,rt) \geq \theta_E(0,\infty) - \varepsilon = \theta_{E_r}(0,\infty) - \varepsilon,\ \forall t \geq \frac{1}{2}.$$

We now let τ with $0 < \tau < \frac{1}{2}$ and ε be as in Proposition 30.3 in [11]. We take $t_0 > 2$ and apply Proposition 30.3 in [11], and get that there is a minimal cone T centered at 0 such that

$$\mathrm{dist}(y,T) \leq \tau t_0, \quad \text{for } y \in E_r \cap B(0, t_0 - \tau) \setminus B\left(0, \frac{1}{2} + \tau\right),$$

$$\mathrm{dist}(z,E_r) \leq \tau t_0, \quad \text{for } z \in T \cap B(0, t_0 - \tau) \setminus B\left(0, \frac{1}{2} + \tau\right),$$

$$\left|\mathcal{H}^1(E_r \cap B(y,u)) - \mathcal{H}^1(T \cap B(y,u))\right| \leq \tau t_0 \tag{4.1.3}$$

$$\text{for any } B(y,u) \subset B(0,t_0-\tau) \setminus B\left(0, \frac{1}{2} + \tau\right), \text{ and}$$

$$\left|\mathcal{H}^1(E_r \cap B(0,t)) - \mathcal{H}^1(T \cap B(0,t))\right| \leq \tau t,\ \forall \frac{1}{2} + \tau \leq t \leq t_0 - \tau. \tag{4.1.4}$$

If we put $N(t) = \#\left(\partial B(0,t) \cap E\right)$, then by Lemma 8.10 in [8] or Theorem 3.2.22 in [19],

$$\frac{1}{s}\int_0^s N(t)dt \leq \frac{1}{s}\mathcal{H}^1(E \cap B(0,s)).$$

Combining this with (4.1.4), we can get that, for $\left(\frac{1}{2} + \tau\right) r \leq s \leq (t_0 - \tau)r$,

$$\frac{1}{s}\int_0^s N(t)dt \leq \frac{1}{s}\mathcal{H}^1(T \cap B(0,s)) + \tau \leq 3 + \tau.$$

But t_0 can be chosen arbitrarily large, thus we can find a sequence $\{s_n\}_{n=1}^{\infty}$ such that $s_n \to \infty$ and $N(s_n) < 4$. Since $L_1 \subset E$, we have that $N(s) \geq 2$ for any $s > 0$, thus $2 \leq N(s_n) \leq 3$.

If $N(s) = 2$ for some $s > 0$, by minimality of E, we can get that

$$E \cap B(0,s) = L \cap B(0,s).$$

If $N(s) = 3$ for some $s > 0$, we suppose that

$$E \cap \partial B(0,s) = \{(-s,0), X_s, (s,0)\},\ X = (x_s, y_s).$$

If the points X_s and 0 are not in the same component of $E \cap \overline{B(0,s)}$, then by minimality of E, we can see that X_s is the only point in the component of $E \cap \overline{B(0,s)}$ which contains the point X_s, and

$$E \cap B(0,s) = L \cap B(0,s).$$

If the points X_s and 0 are in the same component of $E \cap \overline{B(0,s)}$, then there is a path in $E \cap \overline{B(0,s)}$ from X_s to 0, we denote it by $\gamma : [0,1] \to E \cap \overline{B(0,s)}$ with $\gamma(0) = X_s$ and $\gamma(1) = 0$. Let

$$v_0 = \inf\{v \in [0,1] \mid \gamma(v) \in L_1\}.$$

Then

$$E' = [X_s, \gamma(v_0)] \cup (E \setminus B(0,s)) \cup (L_1 \cap B(0,s))$$

and

$$E'' = [(x_s, y_s), (x_s, 0)] \cup (E \setminus B(0,s)) \cup (L_1 \cap B(0,s))$$

are competitors of E, and

$$\mathcal{H}^1(E'' \cap B(0,s)) \leq \mathcal{H}^1(E' \cap B(0,s)) \leq \mathcal{H}^1(E \cap B(0,s)).$$

By minimality of E, we get that $E'' = E' = E$. We put $\gamma(v_0) = (t_s, 0)$, then

$$E \cap B(0,s) = (P_{t_s} \cup L_1) \cap B(0,s).$$

If there exists a sequence $\{n_k\}_{k=1}^{\infty}$ such that $N(s_{n_k}) = 2$, then

$$E \cap B(0, s_{n_k}) = L_1 \cap B(0, s_{n_k})$$

for any $k \geq 1$; thus we get that $E = L_1$.

If there exist an integer $n_0 \geq 1$ such that $N(s_n) = 3$ for $n \geq n_0$, then

$$E \cap B(0, s_n) = (P_{t_{s_n}} \cup L_1) \cap B(0, s_n),$$

and $t_{s_n} = t_{s_{n_0}}$ for any $n \geq n_0$. By putting $t = t_{s_{n_0}}$, we get that $E = P_t \cup L_1$. $\square$

4.2 Two dimensional minimal cone with sliding boundary

In this section we consider a simple case in $\mathbb{R}^3$: our domain Ω is a half space, and the boundary L_1 is the plane which is the boundary of Ω. In the domain Ω, we will see what does a sliding minimal cone look like. For simplicity, we assume that

$$\begin{aligned} \Omega &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq 0\}, \\ L_1 &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}. \end{aligned} \tag{4.2.1}$$

Let us refer to paper [8] for the definition of cones of type $\mathbb{Y}$ and $\mathbb{T}$. We say that a cone $Z \subset \Omega$ is of type $\mathbb{P}_+$, if Z is a closed half plane which is perpendicular to L_1 and through 0, i.e. the intersection of Ω with a plane which is through 0 and meets L_1 perpendicularly; similarly we say that a cone $Z \subset \Omega$ is of type $\mathbb{Y}_+$ if it is a intersection of Ω with a cone in $\mathbb{R}^3$ of type $\mathbb{Y}$which is perpendicular to L_1. Recall that a cone Z in $\mathbb{R}^3$ of type $\mathbb{Y}$ is the union of there half planes bounded by a line ℓ, called the spine of Z. Here we say that a cone of type $\mathbb{Y}$ is perpendicular to L_1, if the spine of the cone is perpendicular to L_1. We will check that cones of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ are sliding minimal.

Let Z be a cone of type $\mathbb{T}$, we say that Z is perpendicular to L_1 if the center of Z locates at the origin and $Z \cap \overline{\Omega^c}$ is a cone of type $\mathbb{Y}_+$ in the domain $\overline{\Omega^c}$.

We say that a cone $Z \subset \Omega$ is of type $\mathbb{T}_+$ if it is the intersection of Ω with a cone of type $\mathbb{T}$ which is perpendicular to L_1. In this paper, we do not discuss whether or not a cone of type $\mathbb{T}_+$ is sliding minimal.

A cone $Z \subset \Omega$ is called of type $\mathbb{V}$ is it can be written as $Z = \mathscr{R}(\mathbb{R} \times V_{\alpha,0})$ where $\mathscr{R}$ is a rotation which maps L_1 into L_1, $V_{\alpha,0}$ is cone in a half plane defined as in Section 4.1 and $0 < \alpha < \frac{\pi}{2}$.

Lemma 4.5. *Let Ω, L_1 be as in (4.2.1). If Z is a cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$, then Z is a sliding minimal cone. If $Z = Z' \cup L_1$ and Z' is a sliding minimal cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$, then Z is also a sliding minimal cone.*

Proof. Suppose that Z is of type $\mathbb{P}_+$ or $\mathbb{Y}_+$, which is not sliding minimal. Then there is a competitor of Z, say E, such that

$$\mathcal{H}^2(E \setminus Z) < \mathcal{H}^2(Z \setminus E).$$

Let $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection with respect to the plane L_1. That is, for any $(x_1, x_2, x_3) \in \mathbb{R}^3$, $\sigma(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. Then $\widetilde{E} = E \cup \sigma(E)$ is a competitor of $\widetilde{Z} = Z \cup \sigma(Z)$, and

$$\mathcal{H}^2(\widetilde{E} \setminus \widetilde{Z}) < \mathcal{H}^2(\widetilde{Z} \setminus \widetilde{E}).$$

But we know that $\widetilde{Z}$ is a plane or a cone of type $\mathbb{Y}$, which is minimal in $\mathbb{R}^3$, that gives a contradiction.

Now suppose that $Z = Z' \cup L_1$, where Z' is cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$. Let E be any competitor of Z. Suppose that E coincide with Z out of the ball $B(0, r/2)$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $\pi(x_1, x_2, x_3) = x_3$. By using Lemma 8.10 in [8] or Theorem 3.2.22 in [19], we get that

$$\int_{E \cap B(0,r)} apJ_m\pi(z) d\mathcal{H}^2(z) = \int_{y \in \mathbb{R}} \int_{z \in \pi^{-1}(y)} 1_{B(0,r) \cap E}(z) d\mathcal{H}^1(z) d\mathcal{H}^1(y),$$

where $apJ_m\pi(x)$ is the approximate Jacobian, see [19]. We can check that $apJ_m\pi(z) \leq 1$ for any $z \in E$. Thus

$$\mathcal{H}^2(E \cap B(0,r) \setminus L_1) \geq \int_0^r \int_{z \in \pi^{-1}(y)} 1_{E \cap B(0,r)}(z) d\mathcal{H}^1(z) d\mathcal{H}^1(y).$$

For any $0 < y < r$, $\pi^{-1}(y)$ is a plane, and $Z \cap \pi^{-1}(y)$ is a line or a Y in this plane, so it is minimal in the plane. But $E \cap \pi^{-1}(y)$ coincide with $Z \cap \pi^{-1}(y)$ out of the ball $B(0,1)$, and it is not hard to check that $E \cap \pi^{-1}(y)$ is connected. Thus

$$\begin{aligned}\int_{z \in \pi^{-1}(y)} 1_{E \cap B(0,r)}(z) d\mathcal{H}^1(z) &= \mathcal{H}^1(E \cap B(0,1) \cap \pi^{-1}(y)) \\ &\geq \mathcal{H}^1(Z \cap B(0,1) \cap \pi^{-1}(y)) \\ &= \int_{z \in \pi^{-1}(y)} 1_{Z \cap B(0,r)}(z) d\mathcal{H}^1(z),\end{aligned}$$

hence

$$\mathcal{H}^2(E \cap B(0,r) \setminus L_1) \geq \int_0^r \int_{z \in \pi^{-1}(y)} 1_{Z \cap B(0,r)}(z) d\mathcal{H}^1(z) d\mathcal{H}^1(y).$$

Since $Z = Z' \cap L_1$, and Z' is a cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$, we have that

$$\mathcal{H}^1(Z \cap B(0,r) \setminus L_1) = \int_0^r \int_{z \in \pi^{-1}(y)} 1_{Z \cap B(0,r)}(z) d\mathcal{H}^1(z) d\mathcal{H}^1(y).$$

We get that

$$\mathcal{H}^2(E \cap B(0,r) \setminus L_1) \geq \mathcal{H}^1(Z \cap B(0,r) \setminus L_1),$$

thus

$$\mathcal{H}^2(E \setminus Z) \leq \mathcal{H}^2(Z \setminus E),$$

and Z is minimal. $\square$

Let Q be any convex polyheddron, x be a point in the interior of Q. If $F \subset Q$ is a compact set with $x \notin F$, then we can find a Lipschitz mapping

$$\Pi_{Q,x} : \mathbb{R}^3 \to \mathbb{R}^3 \tag{4.2.2}$$

such that

$$\Pi_{Q,x}|_{Q^c} = \mathrm{id}_{Q^c}, \ \Pi_{Q,x}(E) \subset \partial Q. \tag{4.2.3}$$

Indeed, we take a very small ball $B(x,r)$ such that $B(x,r) \cap F = \emptyset$, and consider the mapping $\varphi : \mathbb{R}^3 \setminus B(x,r) \to \mathbb{R}^3$ defined by

$$\varphi(y) = \begin{cases} y, & y \in Q^c; \\ \{ty + (1-t)x \mid t \geq 0\} \cap \partial Q, & x \in Q \setminus B(x,r). \end{cases}$$

φ is Lipschitz on $\mathbb{R}^3 \setminus B(x,r)$. By the Kirszbraun's theorem [19, 2.10.43], we can find a Lipschitz mapping $\Pi_{Q,x} : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\Pi_{Q,x}|_{\mathbb{R}^3 \setminus B(x,r)} = \varphi.$$

Lemma 4.6. *Let Ω, L_1 be as in (4.2.1). If Z' is a cone of type $\mathbb{T}_+$, then the cone $Z = L_1 \cup Z'$ is not minimal.*

Here we do not want to talk about whether or not a cone of type $\mathbb{T}_+$ is minimal, it is not so obvious. Recall that a cone of type $\mathbb{T}$ has six faces, that meet by sets of three and with $120°$ angles along four edges (half lines emanating from the center).

Proof. We put $O = (0,0,0)$, $A_1 = (\frac{2\sqrt{2}}{3}, 0, 0)$, $A_2 = (-\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3}, 0)$, $A_3 = (-\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3}, 0)$, $B_1 = (\frac{2\sqrt{2}}{3}, 0, \frac{1}{3})$, $B_2 = (-\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3}, \frac{1}{3})$, $B_3 = (-\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3}, \frac{1}{3})$. We denote by C the triangular prism $A_1A_2A_3B_1B_2B_3$, by Γ the union of eight edges of C. Without loss of generality, we assume that

$$Z = \bigcup_{t \geq 0} t\Gamma.$$

We denote by F_0, F_1, F_2 and F_3 the faces $A_1A_2A_3$, $A_3A_1B_1B_3$, $A_1A_2B_2B_1$ and $A_2A_3B_3B_2$ of the prism C respectively. Consider $\tilde{Z} = (Z \setminus C) \cup F_0 \cup F_1 \cup F_2 \cup F_3$. We will show that $\tilde{Z}$ is a competitor of Z.

We take $x_0 = (0,0,\frac{1}{4})$, then x_0 is in the interior of the triangular prism. We take a Lipschitz mapping Π_{C,x_0} as in (4.2.2). Then $\tilde{Z} = \Pi_{C,x_0}(Z)$ is a competitor of Z.

Since $Z \cap C$ consists of faces (triangles) $A_1A_2A_3$, OA_1B_1, OA_2B_2, OA_3B_3, OB_1B_2, OB_2B_3 and OB_3B_1. By a simple calculation, we can get that

$$\mathcal{H}^2(Z \cap C) = \frac{4\sqrt{2} + \sqrt{3}}{3}.$$

Similarly, $\cap C$ consists of faces F_0, F_1, F_2 and F_3, thus

$$\mathcal{H}^2(\tilde{Z} \cap C) = \frac{2\sqrt{6} + \sqrt{3}}{3}.$$

Therefore

$$\mathcal{H}^2(\tilde{Z} \setminus Z) < \mathcal{H}^2(Z \setminus \tilde{Z}),$$

and Z is not minimal.

$\square$

Definition 4.7 ([31, Definition 2.1])**.** Let B_0 be a closed subset of $\mathbb{R}^n$. Let $\delta, c, \alpha > 0$ be given. We say that a nonempty bounded subset $S \subset \mathbb{R}^n \setminus B_0$ is d-dimensional $(\mathbf{M}, cr^\alpha, \delta)$-minimal relative to B_0 if

$$\mathcal{H}^d(S) < \infty, \; S = \operatorname{supp}(\mathcal{H}^d \llcorner S) \setminus B_0,$$

and

$$\mathcal{H}^d(S \cap W) \leq (1 + cr^\alpha)\mathcal{H}^d(\varphi(S \cap W))$$

whenever $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz with $\operatorname{diam}(W \cup \varphi(W)) = r < \delta$ and $\operatorname{dist}(W \cup \varphi(W), B_0) > 0$, where $W = \{z \in \mathbb{R}^n \mid \varphi(z) \neq z\}$.

When $B_0 = L_1$, $h(r) = cr^\alpha$, $E \subset \Omega$ is a reduced bounded (U, h)-A_+-sliding-almost-minimal set, then it is very easy to see that E is also $(\mathbf{M}, cr^\alpha, \delta)$-minimal relative to B_0. With help of this property, we can use a result of Morgan [31, Regularity Theorem 3.8], which is stated as follows.

Theorem 4.8. *Fix $\delta, c, \alpha > 0$. Let B_0 be a closed subset of $\mathbb{R}^n$. Let S be a one-dimensional $(\mathbf{M}, cr^\alpha, \delta)$-minimal set with respect to B_0. Then S consists of $C^{1,\alpha/2}$ curves that can only meet in three at isolated points of $\mathbb{R}^n \setminus B_0$ and with $120°$ angles .*

Let E be a sliding minimal cone in Ω. Set $K = \partial B(0,1) \cap E$, $S = K \setminus L_1$. We want to show that S is $(\mathbf{M}, cr^\alpha, \delta)$-minimal with respect to $B_0 = L_1$ for some $\alpha, c, \delta > 0$.

Proposition 4.9. *Let Ω, L_1 be as in (4.2.1), $B_0 = L_1$. Let E be a reduced sliding minimal cone in Ω, $K = \partial B(0,1) \cap E$. If $K \setminus L_1 \neq \emptyset$, then K is A_+-sliding-almost-minimal for some gauge function h such that $h(r) = cr$ for $r < \frac{1}{100}$.*

Proof. Let $\{\varphi_t\}_{0 \leq t \leq 1}$ be a deformation with $\operatorname{diam}(\widehat{W}) = r < \frac{1}{100}$, where $\widehat{W}$ as in (4.0.1). If $\widehat{W} \cap K = \emptyset$, we have nothing to prove. We now suppose that $\widehat{W} \cap K \neq \emptyset$; we can find a point $x_0 \in S$, such that $\widehat{W} \subset B(x_0, r)$.

We consider the Lipschitz function $\phi : \mathbb{R} \to [0,1]$ defined by

$$\phi(t) = \begin{cases} 0, & t \leq \frac{1}{4} \\ 4(t - \frac{1}{4}), & \frac{1}{4} < t \leq \frac{1}{2} \\ 1, & \frac{1}{2} < t \leq 2 \\ -4(t-2) + 1, & 2 < t \leq \frac{9}{4} \\ 0, & t > \frac{9}{4}. \end{cases}$$

We consider $\pi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\pi(x) = (1 - \phi(|x|))\, x + \phi(|x|) \frac{x}{|x|};$$

when $|x| \leq \frac{1}{4}$ or $|x| \geq \frac{9}{4}$, $\pi(x) = x$; when $\frac{1}{2} \leq |x| \leq 2$, $\pi(x) = \frac{x}{|x|}$. Also π is a Lipschitz map with

$$\operatorname{Lip}\left(\pi|_{B(x_0,r)}\right) \leq \frac{1}{1-r}. \tag{4.2.4}$$

We put $\widetilde{\varphi} = \pi \circ \varphi_1$; then

$$\widetilde{\varphi}(\partial B(0,1) \cap \Omega) \subset \partial B(0,1) \cap \Omega.$$

For $\varepsilon > 0$ small, we consider the Lipschitz map ψ_ε defined by

$$\psi_\varepsilon(x) = \left(1 - \widetilde{\phi}_\varepsilon(|x|)\right) x + \widetilde{\phi}_\varepsilon(|x|)\, |x|\, \widetilde{\varphi}\left(\frac{x}{|x|}\right),$$

where $\widetilde{\phi}_\varepsilon : \mathbb{R} \to [0,1]$ given by

$$\widetilde{\phi}_\varepsilon(t) = \begin{cases} 1, & t \leq 1 \\ -\frac{1}{\varepsilon}(t-1) + 1, & 1 < t \leq 1 + \varepsilon \\ 0, & t \geq 1 + \varepsilon. \end{cases}$$

It is clear that $\psi_\varepsilon(x) = x$ for $|x| \geq 1 + \varepsilon$, $\psi_\varepsilon(x) = |x|\, \widetilde{\varphi}\left(\frac{x}{|x|}\right)$ for $|x| \leq 1$.

We consider the map $\widetilde{\pi} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\widetilde{\pi}(x) = \begin{cases} x, & 0 \leq |x| \leq 1 \\ \frac{x}{|x|}, & 1 < |x| \leq 2 \\ (2\,|x| - 3)\frac{x}{|x|}, & 2 < x \leq 3 \\ x, & |x| \geq 3, \end{cases}$$

it is Lipschitz, thus the map $\widetilde{\psi_\varepsilon} := \widetilde{\pi} \circ \psi_\varepsilon$ is also Lipschitz. It is easy to see that $\widetilde{E} := \widetilde{\psi}_\varepsilon(E)$ is a competitor of E, because that $\{\varphi'_t\}_{0 \leq t \leq 1}$ defined by $\varphi'_t = (1-t)\mathrm{id} + t\widetilde{\psi_\varepsilon}$ is a deformation. We will compare $\widetilde{E}$ with E. Since E is a cone, $\widetilde{\pi}(x)$ lie in the line through 0 and x, $\widetilde{\pi}$ is the radial projection into the sphere $\partial B(0,1)$ on the annulus $1 \leq |x| \leq 2$, and $\widetilde{\psi}_\varepsilon$ is identity out of the ball $B(0, 1+\varepsilon)$, we can get that $\widetilde{E}$ and E coincide out of the ball $\overline{B(0,1)}$. Since E is minimal, we have that

$$\mathcal{H}^2(E \cap \overline{B(0,1)}) \leq \mathcal{H}^2(\widetilde{E} \cap \overline{B(0,1)}). \tag{4.2.5}$$

Recall that $\widehat{W} \subset B(x_0, r)$; if we put

$$R = \operatorname{diam}((W_1 \cap K) \cup \widetilde{\varphi}(W_1 \cap K)), \tag{4.2.6}$$

where $W_1 = \{x \mid \varphi_1(x) \neq x\}$, then $R \leq r$, thus on the sphere $\partial B(0,1)$, we can see that $\widetilde{E}$ and E coincide out of $B(x_0, R)$, thus we can easily get that

$$\mathcal{H}^2(\widetilde{E} \cap \partial B(0,1)) = \mathcal{H}^2(\widetilde{E} \cap \partial B(0,1) \cap B(x_0, R)) \leq 4\pi R^2. \tag{4.2.7}$$

Applying Theorem 3.2.22 in [19], we have

$$\begin{aligned}
\mathcal{H}^2(E \cap \overline{B(0,1)}) &= \mathcal{H}^2(E \cap B(0,1)) + \mathcal{H}^2(E \cap \partial B(0,1)) \\
&= \int_0^1 \mathcal{H}^1(E \cap \partial B(0,t))dt \\
&= \left(\int_0^1 tdt\right) \mathcal{H}^1(E \cap \partial B(0,1)) \\
&= \frac{1}{2}\mathcal{H}^1(E \cap \partial B(0,1)).
\end{aligned} \tag{4.2.8}$$

By the construction of $\widetilde{E}$, we know that $\widetilde{E}$ coincide with a cone in the ball $B(0,1)$, thus the same reason as above, we have

$$\mathcal{H}^2(\widetilde{E} \cap \overline{B(0,1)}) = \frac{1}{2}\mathcal{H}^1(\widetilde{\varphi}(E \cap B(0,1))) + \mathcal{H}^2(\widetilde{E} \cap \partial B(0,1)).$$

We combine this with (4.2.5), (4.2.7) and (4.2.8), and get that

$$\mathcal{H}^1(K) \leq \mathcal{H}^1(\widetilde{\varphi}(K)) + 8\pi R^2.$$

By our construction of $\widetilde{\varphi}$, we have that for any $x \in \partial B(0,1)$, if $\varphi_1(x) = x$, i.e., $x \notin W_1$, then $\widetilde{\varphi}(x) = x$, thus $K \setminus W_1 = \widetilde{\varphi}(K \setminus W_1)$. We get that

$$\begin{aligned}
\mathcal{H}^1(K \cap W_1) &= \mathcal{H}^1(K) - \mathcal{H}^1(K \setminus W_1) \\
&\leq \mathcal{H}^1(\widetilde{\varphi}(K)) - \mathcal{H}^1(K \setminus W_1) + 8\pi R^2 \\
&\leq \mathcal{H}^1(\widetilde{\varphi}(K \cap W_1)) + 8\pi R^2.
\end{aligned} \tag{4.2.9}$$

Since $R = \operatorname{diam}((W_1 \cap K) \cup \widetilde{\varphi}(W_1 \cap K))$, we can show that

$$\mathcal{H}^1(K \cap W_1) + \mathcal{H}^1(\widetilde{\varphi}(K \cap W_1)) \geq R,$$

combine this with (4.2.9), and get that

$$\mathcal{H}^1(K \cap W_1) \leq \frac{1 + 8\pi R}{1 - 8\pi R}\mathcal{H}^1(\widetilde{\varphi}(K \cap W_1)) \leq (1 + 100R)\mathcal{H}^1(\widetilde{\varphi}(K \cap W_1)),$$

and by (4.2.6) and (4.2.4),

$$\mathcal{H}^1(K \cap W_1) \leq \frac{1+100R}{1-r}\mathcal{H}^1(\varphi_1(K \cap W_1)) \leq (1+200r)\mathcal{H}^1(\varphi_1(K \cap W_1)),$$

the result immediately follows. $\square$

Proposition 4.10. *Let Ω, L_1 be as in (4.2.1). Let $E \subset \Omega$ be a minimal cone, and set $K = E \cap \partial B(0,1)$. Then K consists of arcs C_i of great circles. These arcs can only meet at their extremities. For each extremity, if it is not in L_1, then it is a common extremity of exactly three arcs which meet with $120°$ angles.*

A point in $K \setminus L_1$ is called to be a Y-point if it is a common extremity of exactly three curves which meet with $120°$ angles.

Proof. Applying Theorem 4.8 and Proposition 4.9, we can get $\overline{K \setminus L}$ consists of $C^{1,1/2}$ curves, these curves only meet at their extremities. For each extremity, if it is not in L_1, then it is a common extremity of exactly three curves which meet with $120°$. For any point x in the interior of such a curves C_j, by the same proof as in [8, Proposition 14.1], we can get that there is a neighborhood U_x such that $U_x \cap C_j$ is an arc of great circles. From this, we can immediately deduce the result. $\square$

Lemma 4.11. *Let Ω, L_1, E, K be as in the proposition above. For any $x \in L_1$, we denote by Ω_x the half plane through 0 which is perpendicular to L_1 and the straight line joining x and 0. Then, for any point $x \in K \cap L_1$, any blow-up limit of K at x is a sliding minimal cone in Ω_x with sliding boundary $L_x = \Omega_x \cap L_1$.*

The proof of this Lemma is almost the same as in the first part of the proof of Theorem 8.23 in [8].

Proof. Without loss of generality, we assume that $x = (1,0,0)$. Then $\Omega_x = \{(0,x_2,x_3) \mid x_2 \in \mathbb{R}, x_3 \geq 0\}$, $L_x = \{(0,x_2,0) \mid x_2 \in \mathbb{R}\}$. Let $r_k > 0$, $r_k \to 0$. Suppose that

$$\frac{1}{r_k}(K - x) \to Z$$

and

$$\frac{1}{r_k}(E - x) \to F.$$

It is quite easy to see that $Z \subset \Omega_x$ and $Z \subset F$. Theorem 24.13 in [11] says that F is a sliding minimal cone in Ω with sliding boundary L_1. We denote by D the line though the points 0 and x. As in [8], page 140, we can get that

$$F = D \times F^\sharp \text{ where } F^\sharp = F \cap \Omega_x.$$

Similarly, we can get $F^\sharp$ is a sliding minimal cone in Ω_x with sliding boundary L_x. Let us check that $Z = F^\sharp$. It suffices to show that $F^\sharp \subset Z$, since we already know that $Z \subset F^\sharp$. We take any $z \in F^\sharp$, then there exists a sequence $z_k \in E$ such that

$$\frac{z_k - x}{r_k} \to z. \tag{4.2.10}$$

Since $z \in \Omega_x$, and Ω_x is perpendicular to the line D which pass through the points 0 and x, we get that the angles between the line D and the segments which join the pints x and z_k tend to $\frac{\pi}{2}$, i.e.

$$\theta_k = \mathrm{Angle}(z_k - x, D) \to \frac{\pi}{2}.$$

If $z = 0$, by (4.2.10), we get that

$$\frac{|z_k| - 1}{r_k} \to 0. \tag{4.2.11}$$

If $z \neq 0$, we will show that

$$\frac{|z_k| - 1}{|z_k - x|} \to 0. \tag{4.2.12}$$

We put $\gamma_k = \mathrm{Angle}(z_k, x)$. Then $\gamma_k \to 0$. Since $z \neq 0$, we have that $|z_k - x| \neq 0$ and $\gamma_k \neq 0$ for k large. We consider the triangle formed by the vertices 0, x and z_k. We get that $|z_k - x| \geq |z_k| \sin \gamma_k$. Thus

$$\left| \frac{\cos \gamma_k - 1}{|z_k - x|} \right| \leq \frac{1 - \cos \gamma_k}{|z_k| \sin \gamma_k} \to 0.$$

Since

$$\langle x, z_k - x \rangle = |x||z_k - x| \cos \theta_k = |z_k - x| \cos \theta_k$$

and

$$\langle x, z_k - x \rangle = \langle x, z_k \rangle - |x|^2 = |z_k| \cos \gamma_k - 1,$$

we get that

$$|z_k| \cos \gamma_k = 1 + |z_k - x| \cos \theta_k.$$

Hence

$$\frac{|z_k| - 1}{|z_k - x|} = \frac{\cos \theta_k}{\cos \gamma_k} + \frac{1 - \cos \gamma_k}{|z_k - x| \cos \gamma_k} \to 0.$$

In the case $z \neq 0$, we get, from (4.2.10) and (4.2.12), that

$$\frac{|z_k| - 1}{r_k} \to 0. \tag{4.2.13}$$

Thus, from (4.2.11) and (4.2.13), we get that

$$\frac{1}{r_k}\left(\frac{z_k}{|z_k|}-x\right)=\frac{1-|z_k|}{r_k}\cdot\frac{z_k}{|z_k|}+\frac{z_k-x}{r_k}\to z.$$

But we know that $\frac{z_k}{|z_k|}\in K$, thus $z\in Z$.

□

Lemma 4.12. *Let Ω, L_1, E be as in the proposition above, $S=K\setminus L_1$. For any $x\in\overline{S}\cap L_1$, there is a radius $r>0$ such that there is no Y-point in $S\cap B(x,r)$. Moreover, if a blow-up limit of K at x is a cone $V_{\beta,0}$ for some $\beta\in(0,\frac{\pi}{6}]$, then $K\cap B(x,r)$ is a union of two arcs of great circles meeting at x; in the other cases, $S\cap B(x,r)$ is an arc of great circle which perpendicular to L_1.*

Proof. We will prove that there ais only finite number of Y points in S. We denote $\mathbb{S}^+=\{(x_1,x_2,x_3)\in\mathbb{S}^2\mid x_3>0\}$. Let A be a connected component of $\mathbb{S}^+\setminus S$, $\overline{A}$ be the closure of A

If $\overline{A}\cap L_1=\emptyset$, then $\overline{A}$ is a convex. Indeed, we get, from Proposition 4.10, that each Y-point in S must connect three arcs, these three arcs meet with $120°$. Thus at each corner of ∂A, the interior angle of ∂A at this point must have be $120°$, and $\overline{A}$ must be convex.

Now, if $\overline{A}\cap L_1\neq\emptyset$, $\overline{A}$ is also convex. For the same reason, if the vertex of a corner of ∂A is contained in $\mathbb{S}^+$, then the interior angle of ∂A at this point must have be $120°$. If the vertex of a corner of ∂A is contained in L_1, then the the interior angle of ∂A at this point is no more than $180°$. Thus $\overline{A}$ must be convex.

The number corners in ∂A must be finite. Indeed, there are at most four corners which touch the boundary L_1, because A is convex. If there are infintely many corners in ∂A, then we can very easily to find 8 corners, saying at points $B_1, B_2, \ldots, B_8$, such that these 8 points are contained in S, and the geodesic connecting B_i and B_{i+1} is contained in ∂A, $i=1,2,\ldots,7$. We now consider the convex spherical polygon $B_1B_2\cdots B_8$. By using the Gauss-Bonnet theorem, for example see [5, Theorem V.2.7], we get that

$$\alpha_1+\alpha_2+\frac{\pi}{3}\times 6+\operatorname{Area}(\overline{A})=2\pi, \tag{4.2.14}$$

where α_1 and α_2 are the exterior angle of the corners of $\partial\overline{A}$ at point B_1 and B_8 respectivly. But that is impossible, the equation (4.2.14) gives an absurdity.

If $\overline{A}$ are contained in $\mathbb{S}^+$, we assume that ∂A has n cornes, then Gauss-Bonnet theorem says that

$$\frac{n\pi}{3}+\operatorname{Area}(\overline{A})=2\pi,$$

thus $n < 6$, and Area $\geq \frac{\pi}{3}$. Since the totall area of $\Omega \cap \partial B(0,1)$ is π, there are at most 6 such connected components. Thus there is only a finite number of Y-point in S; otherwise, it should be infintely many connected component of $\mathbb{S}^+ \setminus S$ such that its corners does not touch L_1.

Since there is only a finite number of Y-point in S, we get that for any $x \in \overline{S} \cap L_1$, there is a radius $r_x > 0$ such that there is no Y-point in $S \cap B(x, r_x)$.

Since K is sliding almost minimal, any blow-up limit of K at x is a sliding minimal cone, denote by Z, and

$$\theta_K(x) = \mathcal{H}^1(Z \cap B(0,1)).$$

If Z is a cone like $V_{\beta,0}$ for some $\beta \in (0, \frac{\pi}{6}]$, then $K \cap B(0, r_x)$ must be two arcs, each of these two arcs is a part of a great cicle, and these two arcs meet at x with angle $\pi - 2\beta$. If Z is a half line perpendicular to L_1, then $K \cap B(0, r_x)$ is an arc which is a part of a great cicle, perpendicular to L_1 and through x. If Z is the union of a line in L_1 and a half line which is perpendicular to L_1, then K is the union of $B(0, r_x) \cap \{(x_1, x_2, 0) \mid x_1^2 + x_2^2 = 1\}$ and an arc which is a part of a great cicle, perpendicular to L_1 and through x. $\square$

Lemma 4.13. *Let Ω, L_1 be as in (4.2.1). Let $E \subset \Omega$ be a sliding minimal cone, $K = E \cap \partial B(0,1)$, $S = K \setminus L_1$. Suppose that for each $x \in \overline{S} \cap L_1$, there is a radius $r > 0$ such that $B(x,r) \cap S$ is an arc of a great circle which is perpendicular to L_1. Then there are only there possible kinds of S, that is, $\overline{S} = Z \cap \partial B(0,1)$, where Z is a sliding minimal cone of type of one of $\mathbb{P}_+$, $\mathbb{Y}_+$ and $\mathbb{T}_+$. And hence, $E = Z$ or $E = Z' \cup L_1$ where Z' is a cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$.*

Proof. We put $\mathbb{S}^+ = \Omega \cap \partial B(0,1) \setminus L_1$. Let A be a connected component of $\mathbb{S}^+ \setminus K$, $\overline{A}$ be the closure of A. By Proposition 4.10 and Lemma 4.12, the boundary of $\overline{A}$ is a spherical polygon whose sides are geodesics of the unit sphere. Using the Gauss-Bonnet theorem, see [5, Theorem V.2.7], we get that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n + \mathrm{Area}(\overline{A}) = 2\pi, \tag{4.2.15}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the exterior angle of the corners of $\partial\overline{A}$. From Lemma 4.12 and Proposition 4.10, we can see that, if a corner touch L_1, then, in the situation of this lemma, the corresponding exterior angle must be $\frac{\pi}{2}$; if a corner do not touch L_1, the corresponding exterior angle must be always $\frac{\pi}{3}$. It is quite clear that there are at least two corners on $\partial\overline{A}$, and it cannot happen that there only one corner touching L_1.

We now consider the equation (4.2.15). If $n = 2$, then the two corners must touch L_1, thus $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, and Aera$(\overline{A}) = \pi$. $\overline{A}$ is a quarter of unity sphere.

Let us split $n = 3$ into two cases. If there is no corner touching L_1, then $\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{3}$, and $\overline{A}$ is an equilateral polar triangle with $\mathrm{Area}(A) = \pi$. If it has corner on $\partial\overline{A}$, then there are at least two, thus $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, $\alpha_3 = \frac{\pi}{3}$, and $\overline{A}$ is a isosceles polar triangle with $\mathrm{Area}(A) = \frac{2\pi}{3}$. More precisely, the base of $\overline{A}$ is an arc contained in $L_1 \cap \partial B(0,1)$ with length $\frac{2\pi}{3}$, the vertex opposite to the base is the point $(0,0,1)$.

Similarly, for $n = 4$, we can get two kinds of spherical quadrilaterals, one is spherical quadrilaterals with equal angles $\frac{2\pi}{3}$ and with area $\frac{2\pi}{3}$, another one is spherical quadrilaterals with one side contained in $L_1 \cap \partial B(0,1)$ and with area $\frac{\pi}{3}$.

We can easily see that n can not be larger than 5; otherwise, we can deduce from (4.2.15) that $\mathrm{Area}(\overline{A}) \leq 0$, which is impossible. For the same reason, when $n = 5$, there is only one kind of spherical pentagons. That is, a spherical with all of corners are contained in $\mathbb{S}^+$ and with area $\frac{\pi}{3}$.

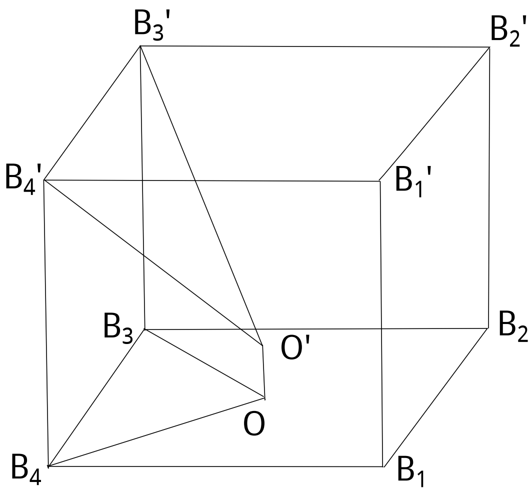
Since each connected component of $\mathbb{S}^+ \cap \partial B(0,1) \setminus K$ has at least area $\frac{\pi}{3}$, and the total area of $\Omega \cap \partial B(0,1)$ is 2π, there are at most six connected component. If there is no Y point on $\mathbb{S}^+$, $E \cap \mathbb{S}^+$ must be a half circle which is contained in $\Omega \cap \partial B(0,1)$ and perpendicular to L_1. Thus $E = Z$ or $E = L_1 \cup Z$, where Z is a cone of type $\mathbb{P}_+$, hence $\overline{S} = Z \cap \partial B(0,1)$.

If there is only one Y point on $\mathbb{S}^+$, then each connect component of $\mathbb{S}^+ \cap \partial B(0,1) \setminus K$ must be a polar triangle with base contained in $L_1 \cap \partial B(0,1)$. By our discussion for $n = 3$, we get that each such connected component is an isosceles polar triangle with area $\frac{2\pi}{3}$. Thus this Y point must be $(0,0,1)$, and $E = Z$ or $E = L_1 \cup Z$, where Z is a cone of type $\mathbb{Y}_+$, hence $\overline{S} = Z \cap \partial B(0,1)$.

If there are two Y points on $\mathbb{S}^+$, then there are at least two polar triangles with base contained in $L_1 \cap \partial B(0,1)$, and the vertices opposite to the bases must be the point $(0,0,1)$, that is impossible.

If there are three Y points on $\mathbb{S}^+$, then these three points must be the vertices of a polar triangle which is contained in $\mathbb{S}^+$, and this triangle is an equilateral polar triangle with area π. Each side of this triangle is a side of spherical quadrilateral, and the opposite side ot this spherical quadrilateral is contained in $L_1 \cap \partial B(0,1)$. In this case $E = Z$ or $E = L_1 \cup Z$, where Z is a cone of type $\mathbb{T}_+$, thus $\overline{S} = Z \cap \partial B(0,1)$.

Since each polar triangle contained in $\mathbb{S}^+$ has area π, there only one such triangle. If there are four Y point in $\mathbb{S}^+$, then these four point in $E \cap \mathbb{S}^+$ must form a spherical quadrilateral, and $\mathbb{S}^+ \setminus K$ consists of five region, that is, five connect component, each of them is a spherical quadrilateral, one of them is contained in $\mathbb{S}^+$, and each of the rest quadrilateral must has one side contained in L_1. Without loss of generality, we suppose that $(1,0,0) \in \overline{S}$, then those four Y points are $(\frac{\sqrt{6}}{3}, 0, \frac{\sqrt{3}}{3})$, $(0, \frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{3})$, $(-\frac{\sqrt{6}}{3}, 0, \frac{\sqrt{3}}{3})$ and $(0, -\frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{3})$, we denote them by B_1', B_2', B_3' and B_4' respectively. In this case, we will show that is impossible. We put $B_1=(\frac{\sqrt{6}}{3},0,0)$, $B_2=(0,\frac{\sqrt{6}}{3},0)$, $B_3=(-\frac{\sqrt{6}}{3},0,0)$ and $B_4=(0,-\frac{\sqrt{6}}{3},0)$, and denote by C the cube $B_1B_2B_3B_4B_1'B_2'B_3'B_4'$, by Γ the union of the edges of cube C. We denote by F_0 the face $B_1'B_2'B_3'B_4'$ of the cube C, denote by F_1 and F_2 the rectangle $B_1B_3B_3'B_1'$ and $B_2B_4B_4'B_2'$ respectively.



We denote by O and O' the points $(0,0,0)$ and $(0,0,\frac{\sqrt{3}-\sqrt{2}}{3})$ respectively, and denote by Q_1, Q_1, Q_3 and Q_4 the polyhedrons $B_1B_2OB_1'B_2'O'$, $B_2B_3OB_2'B_3'O'$, $B_3B_4OB_3'B_4'O'$ and $B_4B_1OB_4'B_1'O'$ respectively. We now take points x_1, x_2, x_3 and x_4 in the interior of pyramids $OB_1B_2B_2'B_1'$, $OB_2B_3B_3'B_2'$, $OB_3B_4B_4'B_3'$ and $OB_4B_1B_1'B_4'$ respectively.

Let mappings Π_{Q_1,x_1}, Π_{Q_2,x_2}, Π_{Q_3,x_3} and Π_{Q_4,x_4} be as in (4.2.2). We take $\psi=\Pi_{Q_4,x_4}\circ\Pi_{Q_3,x_3}\circ\Pi_{Q_2,x_2}\circ\Pi_{Q_1,x_1}$. Then ψ is a Lipschitz mapping. We now consider the competitor $\tilde{E}=\psi(E)$ of E.

We denote $B_5=B_1$. By a simple calculation, we can get that

$$\begin{aligned}\mathcal{H}^2(E\setminus\tilde{E})-\mathcal{H}^2(\tilde{E}\setminus E)&=\sum_{i=1}^{4}|OB_i'B_{i+1}'|-\left(\sum_{i=1}^{4}O'B_i'B_{i+1}'+\sum_{i=1}^{4}|OO'B_i'|\right)\\&=\frac{2\sqrt{2}+4\sqrt{3}-4}{3}>0,\end{aligned}$$

that is contradict to minimality of E.

It could not happen that there are at least six Y point in $\mathbb{S}^+$, because otherwise, there will be at least six spherical quadrilaterals which touch L_1, but we know that such a quadrilateral has area $\frac{\pi}{3}$, and total area of $\mathbb{S}^+$ is 2π, that is impossible. If there are five Y point in $\mathbb{S}^+$, similar to above case, these five point form a spherical pentagon. By the same techniques used for the above case, we can prove that this is impossible. $\square$

Lemma 4.14. *Let Ω, L_1 be as in (4.2.1). Let $E \subset \Omega$ be a sliding minimal cone, $K = E \cap \partial B(0,1)$. If there exists a point $x_0 \in K \cap L_1$ such that a blow-up limit of K at x_0 is a sliding minimal cone $V_{\alpha,0}$ for some $\alpha \in (0, \frac{\pi}{6}]$, then E is a cone of type $\mathbb{V}$.*

Proof. By Lemma 4.12, there exists a radius $r > 0$ such that $S \cap B(x_0, r)$ is a union of two arcs. That is, $E \cap B(x_0, r) = Z \cap B(x_0, r)$ where Z is a cone of type $\mathbb{V}$. Without loss of generality, we assume $x_0 = (1, 0, 0)$.

We denote $\mathbb{S}^+ = \Omega \cap \partial B(0,1) \setminus L_1$. Let A be a connected component of $\mathbb{S}^+ \setminus K$ which contains a corner with interior angle α. Using the Gauss-Bonnet Theorem, see [5, Theorem V.2.7], we get that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n + \mathrm{Area}(\overline{A}) = 2\pi, \tag{4.2.16}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the exterior angle of the corners of $\partial\overline{A}$. We know that there are two corners which touch the boundary L_1, assume that $\alpha_1 = \pi - \alpha$ and α_2 are the corresponding exterior angles. It is quite easy to see that A is contained in a spherical lune enclosed by two great circles with angle α, thus $\mathrm{Area}(\overline{A}) \leq 2\alpha$.

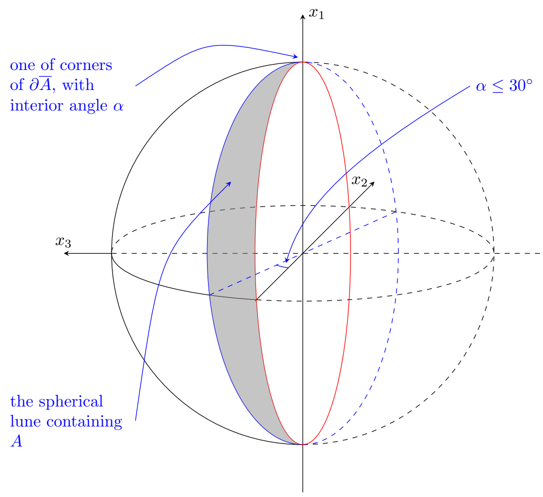


Figure 4.1: spherical lune

If $\alpha_2 = \frac{\pi}{2}$, then $\text{Area}(\overline{A}) < 2\alpha$,

$$(\pi - \alpha) + \frac{\pi}{2} + \frac{\pi}{3} \times (n-2) + \text{Area}(\overline{A}) = 2\pi,$$

thus

$$\frac{\pi}{2} - \alpha < \frac{n-2}{3}\pi < \frac{\pi}{2} + \alpha.$$

Since $\alpha \in (0, \frac{\pi}{6}]$, we get that $3 < n < 4$, that is imppossible.

If $\alpha_2 \neq \frac{\pi}{2}$, then $\alpha_2 \geq \frac{5\pi}{6}$, thus

$$2\pi = (\pi - \alpha) + \alpha_2 + \frac{(n-2)\pi}{3} + \text{Area}(\overline{A}) > (\pi - \alpha) + \frac{5\pi}{6} + \frac{(n-2)\pi}{3}.$$

Since $\alpha \leq \frac{\pi}{6}$, we get that $2 \leq n < 3$, hence $n = 2$. In this case, A must be a spherical lune enclosed by two great circles with angle α, and $E = \mathbb{R} \times V_{\alpha,0}$. □

Theorem 4.15. *Let Ω, L_1 be as in (4.2.1). Let $E \subset \Omega$ be a sliding minimal cone. If $L_1 \subset E$ and $E \setminus L_1 \neq \emptyset$. Then $E = Z \cup L_1$, Z is a cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$.*

Proof. The result immediately follows from Lemma 4.12, Lemma 4.13 and Lemma 4.14. Indeed, by putting $K = E \cap \partial B(0,1)$ and $S = K \setminus L_1$, Lemma 4.11 says that any blow-up limit of K at a point $x \in K \cap L_1$ is a one dimensional sliding minimal cone. By Lemma 4.3, there exist only there possible cases for such a minimal cone. That is, a half line P_0 perpendicular to L_1, or P_0 union the line which pass though x perpendicular the segment $[0,x]$ and is contained in L_1, or a cone $V_{\alpha,0}$. If it is a cone $V_{\alpha,0}$, then by Lemma 4.14, $E = \mathbb{R} \times V_{\alpha,0}$, which is impossible. For any $x \in \overline{S} \cap L_1$, by Lemma 4.12, there exists a radius $r > 0$ such that $S \cap B(x,r)$ is an arc of a great cicle which is perpendicular to L_1, and by Lemma 4.13, $E = Z \cup L_1$, where Z is a cone of type of one of $\mathbb{P}_+$, $\mathbb{Y}_+$ and $\mathbb{T}_+$, but for the last case, it is impossible, because we know that $Z \cup L_1$ is not minimal when Z is of type $\mathbb{T}_+$. We get that $E = Z \cup L_1$, where Z is a cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$. □

Remark 4.16. *We claim that the list of sliding minimal cones is the following: cones of type $\mathbb{P}_+$, cones of type $\mathbb{Y}_+$, the plane L_1, cones like $\mathscr{R}(\mathbb{R} \times V_{\alpha,0})$ with $0 < \alpha \leq \frac{\pi}{6}$, cones $L_1 \cup Z$ where Z are cones of type $\mathbb{P}_+$ or $\mathbb{Y}_+$, and cones type $\mathbb{T}_+$.*

We did not prove that a cone of type $\mathbb{T}_+$ is sliding minimal. Indeed it can probably be proved by using calibration, but this may take us too much time, we do not want to do it here. It is also not too hard to check that a cone like

$\mathbb{R} \times V_{\alpha,0}$ is sliding minimal if and only if $0 < \alpha \leq \frac{\pi}{6}$. One of possible ways to do this is to sue almost the same technique as in Lemma 4.5, but again we omit it. In fact, we do not need know whether or not a cone of type $\mathbb{T}_+$ or like $\mathbb{R} \times V_{\alpha,0}$ is minimal in this paper. For the rest in the list, we know from Lemma 4.12, Lemma 4.13 and Lemma 4.14 that they are sliding minimal.

4.3 Reifenberg's theorem

We want to use a result, Theorem 2.2 in [13]. But here we are in the half space, the theorem can not be used directly, it should be adapted a little bit.

Let n and d be two integers with $2 \leq d < n$. We take

$$\begin{aligned} \Omega_n &= \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}, \\ L_1 &= \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_n = 0\}. \end{aligned} \tag{4.3.1}$$

We let σ be the reflection with respect to L_1. That is, the function $\mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\sigma(x_1, \cdots, x_{n-1}, x_n) = (x_1, \cdots, x_{n-1}, -x_n).$$

Let TG be a class of sets defined as in [13, p.6], which consists of 3 kinds of cones (centered at any point in $\mathbb{R}^n$) of dimension d in $\mathbb{R}^n$. In particular, if $n = 3$, $d = 2$, TG consists of planes, cones which are the union of three half planes bounded by a line while the angle between any two half planes is larger than a constant $\tau_0 > 0$ (they look like cones of type $\mathbb{Y}$), and cones which are union of several faces that meet only by sets of three and with angles between two adjacent faces and angles between the spines larger than a constant $\tau_0 > 0$ (cones of type $\mathbb{T}$ and cones look like of type $\mathbb{T}$ are such cones; cone $Z \cup \sigma(Z)$ is also a such cone, where Z is a cone of type $\mathbb{Y}_+$ or $\mathbb{T}_+$. Of course, far more than these cones).

For any $x \in \mathbb{R}^n$, $r > 0$, we will consider $d_{x,r}$ a variant of the Hausdorff distance on closed sets, which is defined by

$$d_{x,r}(E, F) = \frac{1}{r} \max \left\{ \sup_{z \in E \cap B(x,r)} \operatorname{dist}(z, F), \sup_{z \in F \cap B(x,r)} \operatorname{dist}(z, E) \right\}.$$

If E and F are two cones centered at x, then $d_{x,r}(E, F) = d_{x,1}(E, F)$ for any $r > 0$.

Theorem 4.17. *Let $E \subset \mathbb{R}^n$ be a compact set that contains origin with $\sigma(E) = E$, and suppose that for each $x \in E \cap B(0, 2)$ and each ball $B(x, r) \subset B(0, 2)$, we can find $Z(x, r) \in TG$ that contains x, such that*

$$d_{x,r}(E, Z(x, r)) \leq \varepsilon.$$

Suppose, additionally, that $Z(\sigma(x), r) = \sigma(Z(x, r))$. If $\varepsilon > 0$ is small enough, depending only on n, d and τ_0, then there exist a cone $Z \in TG$ centered at origin and a mapping $f : B(0, 3/2) \to B(0, 2)$ with the following properties:

$$
\begin{gathered}
\sigma(Z) = Z,\ f \circ \sigma = \sigma \circ f,\ \|f - \mathrm{id}\|_\infty \leq \alpha, \\
(1+\alpha)^{-1} |x - y|^{1+\alpha} \leq |f(x) - f(y)| \leq (1+\alpha) |x - y|^{\frac{1}{1+\alpha}}, \\
B(0,1) \subset f\left(B\left(0, \frac{3}{2}\right)\right) \subset B(0,2), \\
E \cap B(0,1) \subset f\left(Z \cap B\left(0, \frac{3}{2}\right)\right) \subset E \cap B(0,2),
\end{gathered} \tag{4.3.2}
$$

where α only depends on only n, d, τ_0 and ε, and τ_0 is defined as in [13, (2.7) and (2,8)].

Proof. The proof is essential the same as in [13], we only need do a little change. Here we use same notation as in [13]. We firstly remark that $\sigma(E_i) = E_i$, $i = 1, 2$ or 3, where E_i are defined as in [13, p.11, p.12].

Next, we modify a little the construction of a good covering of E at scale 2^{-n}, that is in Section 5 in [13, Covering and partition of unity]. The first step is just same the as the original construction; if the condition (4.36) in [13] holds, we cover $E_3 \cap B(0, 199/100) = \{0\}$ with the ball $B_{i_0} = B(0, 2^{-n-20})$, and set $I_3 = \{i_0\}$; if the condition (4.35) in [13] holds, we take $I_3 = \emptyset$ and choose no ball. In the second step, for the construction of a covering of

$$E_2' = E_2 \cap B(0, 198/100) \setminus \frac{7}{4} B_{i_0},$$

we modify a little the original construction to adapt to our case. We put $E_2'' = E_2' \cap \Omega_n$, then $E_2' = E_2'' \cap \sigma(E_2'')$. Select a maximal subset X_2'' of E_2'', with the property that different points of X_2'' have distances at least 2^{-n-40}. We put $X_2 = X_2'' \cup \sigma(X_2'')$, and for accounting reasons, we suppose that $X_2'' = \{x_i\}_{i \in I_2''}$, $I_2'' \cap I_3 = \emptyset$, and that $X_2' = \sigma(X_2'') = \{x_i\}_{i \in I_2'}$, $I_2' \cap (I_2'' \cup I_3) = \emptyset$. Let $I_2 = I_2' \cup I_2''$, $X_2 = X_2' \cup X_2''$. We put $r_i = 2^{-n-40}$ and $B_i = B(x_i, r_i)$ for $i \in I_2$. We can see that the balls $\overline{B_i}$, $i \in I_2$, cover E_2'. In the third step, we put

$$V_2 = \frac{15}{8} B_{i_0} \cup \bigcup_{i \in I_2} \frac{7}{4} B_i \text{ and } E' = E_1 \cap B\left(0, \frac{197}{100}\right) \setminus V_2.$$

Similarly to the above step, put $E_1'' = E_1' \cap \Omega_n$, and select a maximal subset X_1'' of E_1'', with the property that different points of X_1'' have distances at least 2^{-n-60}, and then suppose that $X_1'' = \{x_i\}_{i \in I_1''}$ with $I_1 \cap (I_2 \cup I_3) = \emptyset$, and that $X_1' = \sigma(X_1'') = \{x_i\}_{i \in I_1'}$ with $I_1' \cap (I_1'' \cup I_2 \cup I_3)$. Set $I_1 = I_1' \cup I_1''$, and

$B_i = B(x_i, 2^{-n-60})$ for $i \in I_1$. It is very easy to see that the balls $\overline{B_i}$, $i \in I_1$, cover E'_1. For the fourth and last step of the construction of the covering, we put

$$V_1 = \frac{31}{16} B_{i_0} \cup \bigcup_{i \in I_2} \frac{15}{8} B_i \cup \bigcup_{i \in I_1} \frac{7}{4} B_i \text{ and } E'_0 = \mathbb{R}^3 \setminus V_1.$$

We put $E''_0 = E'_0 \cap \Omega_n$, and pick a maximal subset X''_0 of E''_0, such that different points of X''_0 have distance at least 2^{-n-80}, and then suppose that $X''_0 = \{x_i\}_{i \in I''_0}$ with $I''_0 \cap (I_1 \cup I_2 \cup I_3) = \emptyset$, and that $X'_0 = \{x_i\}_{i \in I'_0}$ with $I'_0 \cap (I''_0 \cup I_1 \cup I_2 \cup I_3) = \emptyset$. Set $I_0 = I'_0 \cup I''_0$, and $B_i = B(x_i, 2^{-n-80})$ for $i \in I_0$, then the balls $\overline{B_i}$, $i \in I_0$, cover E'_0.

For the selection of a partition of unity in equation (5.10) in [13], we choose the $\widetilde{\theta}_i$ as the translation and dilation of a same model θ, where θ is a smooth function such that $\theta(x) = 1$ in $B(0,2)$, $\theta(x) = 0$ out of $B(0,3)$, $0 \le \theta(x) \le 1$ everywhere, and $\sigma \circ \theta = \theta \circ \sigma$. The rest of proof will be the same as in [13].

We now verify that $\sigma \circ f^* = f^* \circ \sigma$. It is clear that $\sigma \circ f_0^* = f_0^* \circ \sigma$, $\sigma \circ f_0 = f_0 \circ \sigma$, $\sigma \circ \psi_{i_0}^* = \psi_{i_0}^* \circ \sigma$, and $\sigma \circ \psi_{i_0} = \psi_{i_0} \circ \sigma$. By our construction of X_0, X_1 and X_2, we can see that

$$\begin{aligned} g_n^*(x) &= \sum_{i \in I_n} \theta_i(x) \psi_i^*(x) \\ &= \theta_{i_0}(x) \psi_{i_0}^*(x) + \sum_{i \in I'_0 \cup I'_1 \cup I'_2} \Big(\theta_i(x) \psi_i^*(x) + \theta_i(\sigma(x)) \psi_i^*(\sigma(x)) \Big), \end{aligned}$$

thus $\sigma \circ g_n^* = g_n^* \circ \sigma$. By induction on n, we can get that $\sigma \circ f_n^* = f_n^* \circ \sigma$ for all $n \ge 0$. f^* is the limit of the sequence f_n^*, thus $\sigma \circ f^* = f^* \circ \sigma$.

Finally, by the same proof as above, we can prove that $\sigma \circ f = f \circ \sigma$. □

Corollary 4.18. *For each small $\tau > 0$, we can find $\varepsilon > 0$, that depends only on n, τ and τ_0 such that if $E \subset \Omega$ is a closed set, $0 \in E$ and $r > 0$ are such that for $y \in E \cap B(0,3r)$ and $0 < t \le 3r$, we can find $Z(y,t)$, which is a minimal cone in $\mathbb{R}^3$ when $0 < t < \operatorname{dist}(y, L_1)$ and a sliding minimal cone in Ω with boundary L_1 when $\operatorname{dist}(y, L_1) \le t \le 3r$, such that*

$$d_{y,t}(E, Z(y,t)) \le \varepsilon,$$

and in addition $Z(0,3r)$ is sliding minimal cone centered at 0, then there is a sliding minimal cone centered at origin and a mapping $f : B(0,3r/2) \cap \Omega \to$

$B(0,2r)\cap\Omega$ *with the following properties:*

$$\begin{gathered} f(x)\in L_1 \text{ for } x\in L_1\cap B(0,3r/2),\ \ \|f-\mathrm{id}\|_\infty\le\tau r,\\ C\left|x-y\right|^{1+\tau}\le\left|f(x)-f(y)\right|\le C^{-1}\left|x-y\right|^{\frac{1}{1+\tau}},\\ B(0,r)\cap\Omega\subset f\left(B\left(0,\frac{3r}{2}\right)\cap\Omega\right)\subset B(0,2r)\cap\Omega,\\ E\cap B(0,r)\subset f\left(Z\cap B\left(0,\frac{3r}{2}\right)\right)\subset E\cap B(0,2r), \end{gathered}\tag{4.3.3}$$

where C is a constant which only depends on τ and r.

4.4 Regularity of sliding almost minimal sets I

In this section, we restrict ourselves to the half space Ω, and prove some boundary regularity for sliding almost minimal sets.

Lemma 4.19. *Let Ω and L_1 be as in (4.2.1), U an open set. Suppose that $E\subset\Omega$ is (U,h)-sliding-almost-minimal. For each $\tau>0$, we can find $\varepsilon(\tau)>0$ such that if $x\in E\cap L_1$ and $r>0$ are such that*

$$B(x,r)\subset U, h(2r)\le\varepsilon(\tau), \int_0^{2r}\frac{h(t)}{t}dt\le\varepsilon(\tau), \theta(x,r)\le\theta(x)+\varepsilon(\tau),\tag{4.4.1}$$

then for every $\rho\in(0,9r/10]$ there is a sliding minimal cone Z_x^ρ centered at x, such that

$$d_{x,\rho}(E,Z_x^\rho)\le\tau\tag{4.4.2}$$

and for any ball $B(y,t)\subset B(x,\rho)$,

$$\left|\mathcal{H}^2(E\cap B(y,t))-\mathcal{H}^2(Z_x^\rho\cap B(y,t))\right|\le\tau\rho^2\tag{4.4.3}$$

Moreover, if $L_1\subset E$, then we can suppose that $L_1\subset Z_x^\rho$.

This lemma is directly following from Proposition 30.19 in [11]. If $L_1\subset E$, by the original proof in [11, Proposition 30.19], we can go further and assert that $L_1\subset Z^\rho$, the proof will be same, we do not even need to do any extra effort.

If E is a sliding almost minimal set, then for any $x\in E\cap L_1$, any blow-up limit of E at x is a sliding minimal cone, see [11, Theorem 24.13]. Moreover, the density of any blow-up limit at origin is aways the value $\theta_E(x)$, see Proposition 7.31 [8] and Corollary 29.53 in [11]. By Remark 4.16,

$$\theta_E(x)\in\left\{\frac{1}{2},\frac{3}{4},1,d_{T_+},\frac{3}{2},\frac{7}{4}\right\},$$

where we denote by d_{T_+} the density of cones of type $\mathbb{T}_+$ at origin. In fact, $d_{T_+} = 3\arccos(-1/3)/\pi - 3/4 \approx 1.07452$.

If $\theta_E(x) = \frac{1}{2}$, then $\theta_{Z_x^\rho}(x) = \frac{1}{2}$, thus Z_x^ρ is a sliding minimal cone of type $\mathbb{P}_+$ in Ω with sliding boundary L_1.

Similarly, if $\theta_E(x) = \frac{3}{4}$, then $\theta_{Z_x^\rho}(x) = \frac{3}{4}$, thus Z_x^ρ is a sliding minimal cone of type $\mathbb{Y}_+$ in Ω with sliding boundary L_1.

If $L_1 \subset E$, then the blow-up limit is sliding minimal cone containing L_1. We know, by Theorem 4.15, that there are only three kinds of minimal cone which contain L_1. That is, L_1 or $Z \cup L_1$, here Z is a minimal cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$. Thus, in the case $L_1 \subset E$, there are three possible values for $\theta_E(x)$, that is 1, $\frac{3}{2}$ and $\frac{7}{4}$. In particular, if $L_1 \subset E$ and $\theta_E(x) = 1$, then $Z_x^\rho = L_1$.

Lemma 4.20. *Let $E \subset \Omega$ be a sliding almost minimal set, $L_1 \subset E$. If a blow-up limit of E at $x \in L_1$ is the plane L_1, then there exists $r > 0$ such that $E \cap B(x,r) = L_1 \cap B(x,r)$.*

Proof. Without loss of generality, we assume $x = 0$. L_1 is a blow-up limit of E at 0. By corollary 29.53 in [11], we get that $\theta_E(0) = 1$. Let $\tau > 0$ be a small enough number, let $\varepsilon(\tau)$ be as in Lemma 4.19. We take $0 < \tau_2 \leq \frac{\varepsilon(\tau)}{2}$, and let $\varepsilon(\tau_2)$ be as in Lemma 4.19. We take $r > 0$ such that

$$(1 + \varepsilon(\tau_2)) \exp\left(\lambda \int_0^{\frac{r}{3}} \frac{h(2t)}{t} dt\right) < \frac{3}{2}$$

and

$$\theta_E(0,r) \leq 1 + \varepsilon(\tau_2),$$

where λ is taken as in Proposition 5.24 in [8].

By lemma 4.19, for any $0 < \rho \leq \frac{9r}{10}$,

$$d_{0,\rho}(E, L_1) \leq \tau_2, \tag{4.4.4}$$

and for all ball $B(y,t) \subset B(0,\rho)$,

$$\left|\mathcal{H}^2(E \cap B(y,t)) - \mathcal{H}^2(L_1 \cap B(y,t))\right| \leq \tau_2 \rho^2. \tag{4.4.5}$$

Thus, in particular, for any $y \in B(0, \frac{r}{3}) \cap E$,

$$\theta_E\left(y, \frac{r}{3}\right) = \left(\pi \left(\frac{r}{3}\right)^2\right)^{-1} \mathcal{H}^2\left(E \cap B\left(y, \frac{r}{3}\right)\right) \leq 1 + \frac{4\tau_2}{\pi}.$$

For any $y \in B(0, \frac{r}{3}) \cap L_1$, by Theorem 28.7 in [11], we get that

$$1 \leq \theta_E(y) \leq \theta_E\left(y, \frac{r}{3}\right) \exp\left(\lambda \int_0^{\frac{r}{3}} \frac{h(2t)}{t} dt\right) < \frac{3}{2},$$

thus

$$\theta_E(y) = 1.$$

Therefore, for $y \in B\left(0, \frac{r}{3}\right) \cap L_1$,

$$\theta_E\left(y, \frac{r}{3}\right) \leq 1 + \frac{4\tau_2}{\pi} \leq \theta_E(y) + \varepsilon(\tau).$$

By lemma 4.19, for any $0 < \rho \leq \frac{3r}{10}$,

$$d_{y,\rho}(E, L_1) \leq \tau. \tag{4.4.6}$$

We shall deduce, from equation (4.4.6), that for any $0 < \rho < \frac{r}{3}$,

$$E \cap B(0, \rho) = L_1 \cap B(0, \rho).$$

Once we have proved this, the desire result follows. We assume, for the sake of a contradiction, that

$$E \cap B(0, \rho) \setminus L_1 \neq \emptyset.$$

Let $z \in E \cap B(0, \rho) \setminus L_1$, and let y be the projection of z on L_1, then $0 < |z - y| < \rho$. We choose ρ' such that

$$|z - y| < \rho' < \min\left\{\rho, \frac{|z - y|}{\tau}\right\}.$$

From equation (4.4.6), we can get that

$$|z - y| \leq \rho' d_{y,\rho'}(E, L_1) \leq \tau\rho' < |z - y|,$$

absurd. □

Lemma 4.21. *Let Ω, L_1 and U be as in Lemma 4.19, let $E \subset \Omega$ be a (U, h)-sliding-almost-minimal set with $L_1 \subset E$. Let $F = \overline{E \setminus L_1}$. Then $\mathcal{H}^2(F \cap L_1) = 0$, and F is also (U, h)-sliding-almost-minimal.*

Proof. We put $G = F \cap L_1$. Let $0 < \varepsilon < 1/100$. We assume, for the sake of contradiction, that $\mathcal{H}^2(G) > 0$. Since G is a subset of L_1, it is rectifiable, thus for $\mathcal{H}^2$-a.e. $x \in G$, $\theta_G(x) = 1$. Without loss of generality, we suppose that $\theta_G(0) = 1$, then there exists a radius $r_1 > 0$ such that for all $0 < r \leq r_1$,

$$\theta_G(0, r) \geq 1 - \varepsilon. \tag{4.4.7}$$

Since E is sliding almost minimal, by Theorem 28.7 (almost monotonicity of density property) in [11], we can find a radius $r_2 > 0$ such that for all $0 < r \leq r_2$,

$$1 - \varepsilon \leq \frac{\theta_E(0, r)}{\theta_E(0)} \leq 1 + \varepsilon. \tag{4.4.8}$$

Since E is sliding almost minimal and $L_1 \subset E$, by Lemma 4.19, there exists $r > 0$ such that for any $0 < \rho \leq r$, there exists a sliding minimal cone $Z_0^\rho \supset L_1$ such that

$$d_{0,\rho}(E, Z_0^\rho) \leq \varepsilon \tag{4.4.9}$$

and for any ball $B(y,t) \subset B(0,\rho)$,

$$\left|\mathcal{H}^2(E \cap B(y,t)) - \mathcal{H}^2(Z_0^\rho \cap B(y,t))\right| \leq \varepsilon\rho^2 \tag{4.4.10}$$

We take $0 < \rho \leq \min\{r, r_1, r_2\}$, and consider a collection of balls

$$\mathcal{V} = \left\{ B(x,s) \;\middle|\; \begin{array}{c} x \in G \cap B(0,\rho), s \leq \varepsilon\rho, B(x,s) \subset B(0,\rho) \\ \theta_G(x,s) \geq 1 - \varepsilon, \theta_E(x,s) \geq (1-\varepsilon)\theta_E(x) \end{array} \right\},$$

it is a Vitali covering for $G \cap B(0,\rho)$. By a Vitali's covering theorem for the Hausdorff measure, see for example, there exists a finite or countably infinite disjoint subcollection $\{B_i\}_{i \in I} \subset \mathcal{V}$ such that

$$\mathcal{H}^2\left(G \cap B(0,\rho) \setminus \bigcup_{i \in I} B_i\right) = 0. \tag{4.4.11}$$

We now consider two balls $B_1' = B(y_1, t_1)$ and $B_1' = B(y_2, t_2)$, where $y_1 = (0, 0, \frac{1+\varepsilon}{2}\rho)$, $y_2 = (0, 0, -\frac{1+\varepsilon}{2}\rho)$ and $t_1 = t_2 = \frac{1-\varepsilon}{2}\rho$. We can see that $B_1' \subset B(0,\rho)$ and $B_2' \subset B(0,\rho)$, thus by equation (4.4.10), we can get that

$$\mathcal{H}^2(E \cap B_1') \geq \mathcal{H}^2(Z_0^\rho \cap B_1') - \varepsilon\rho^2 \tag{4.4.12}$$

and

$$\mathcal{H}^2(E \cap B_2') \geq \mathcal{H}^2(Z_0^\rho \cap B_2') - \varepsilon\rho^2. \tag{4.4.13}$$

It is very easy to see that $\{B_1', B_2'\} \cup \{B_i\}_{i \in I}$ is a family of disjoint balls and

$$B_1' \cup B_2' \cup \bigcup_{i \in I} B_i \subset B(0,\rho). \tag{4.4.14}$$

For $i \in I$, we denote $B_i = B(x_i, s_i)$, then $x_i \in G$ and $\theta_E(x_i) \geq \frac{3}{2}$; otherwise, $\theta_E(x_i) = 1$, any blow-up limit of E at x_i must be L_1, and by Lemma 4.20, there is a small ball $B(x_i, r')$ such that $E \cap B(x_i, r') = L_1 \cap B(x_i, r')$, that is impossible.

By our choice of $\mathcal{V}$, we have that

$$\theta_E(x_i, s_i) \geq (1-\varepsilon)\theta_E(x_i) \geq \frac{3}{2}(1-\varepsilon),$$

thus

$$\mathcal{H}^2(E \cap B_i) \geq \frac{3}{2}(1-\varepsilon)\pi s_i^2 \geq \frac{3}{2}(1-\varepsilon)\mathcal{H}^2(G \cap B_i), \tag{4.4.15}$$

and combine with equations (4.4.11) and (4.4.7), to obtain

$$\begin{aligned}\sum_{i \in I} \mathcal{H}^2(E \cap B_i) &\geq \frac{3}{2}(1-\varepsilon)\sum_{i \in I} \mathcal{H}^2(G \cap B_i)\\ &\geq \frac{3}{2}(1-\varepsilon)\mathcal{H}^2(G \cap B(0,\rho))\\ &\geq \frac{3\pi}{2}(1-\varepsilon)^2\rho^2.\end{aligned} \tag{4.4.16}$$

Since $0 \in G$, we have that $\theta_E(0) \geq \frac{3}{2}$, thus $\theta_E(0) = \frac{3}{2}$ or $\theta_E(0) = \frac{7}{4}$.

If $\theta_E(0) = \frac{3}{2}$, the sliding minimal Z_0^ρ which we chose in (4.4.9) can be written $Z_0^\rho = L_1 \cap Z^\rho$, where Z^ρ is a sliding minimal cone of type $\mathbb{P}_+$. In this case, $Z_0^\rho \cap B_i'$, $i = 1, 2$, are two disks with radius $\frac{1-\varepsilon}{2}\rho$, thus

$$\mathcal{H}^2(Z_0^\rho \cap B_i') = \pi\left(\frac{1-\varepsilon}{2}\rho\right)^2,$$

combine this equation with equations (4.4.12), (4.4.13), (4.4.14) and (4.4.16), we can get that

$$\begin{aligned}\mathcal{H}^2(E \cap B(0,\rho)) &\geq \sum_{i=1}^{2} \mathcal{H}^2(E \cap B_i') + \sum_{i \in I} \mathcal{H}^2(E \cap B_i)\\ &\geq 2\pi\left(\frac{1-\varepsilon}{2}\rho\right)^2 + \frac{3}{2}\pi(1-\varepsilon)^2\rho^2 - 2\varepsilon\rho^2\\ &> (2-5\varepsilon)\pi\rho^2,\end{aligned} \tag{4.4.17}$$

but from equation (4.4.8), we can get that

$$\mathcal{H}^2(E \cap B(0,\rho)) \leq \frac{3}{2}(1+\varepsilon)\pi\rho^2, \tag{4.4.18}$$

which contradict with equation (4.4.17), because $0 < \varepsilon < \frac{1}{100}$.

If $\theta_E(0) = \frac{7}{4}$, a very similar calculation as above case, we can get that

$$\mathcal{H}^2(Z_0^\rho) = 3 \times \frac{\pi}{2}\left(\frac{1-\varepsilon}{2}\rho\right)^2,$$

and

$$\mathcal{H}^2(E\cap B(0,\rho)) \geq \frac{3}{4}(1-\varepsilon)^2\pi\rho^2 + \frac{3}{2}(1-\varepsilon)^2\pi\rho^2 - 2\varepsilon\rho^2 > \left(\frac{9}{4} - \frac{11}{2}\varepsilon\right)\pi\rho^2,$$

but from equation (4.4.8), we obtain that

$$\mathcal{H}^2(E\cap B(0,\rho)) \leq \frac{7}{4}(1+\varepsilon)\pi\rho^2,$$

we also get a contradiction. We proved that $\mathcal{H}^2(F\cap L_1)=0$. We will go to prove that F is also sliding almost minimal.

Let $\{\varphi_t\}_{0\leq t\leq 1}$ be any δ-sliding-deformation. Since E is (U,h)-sliding-almost-minimal, applying Proposition 20.9 in [11], we get that

$$\mathcal{H}^2(E\setminus\varphi_1(E)) \leq \mathcal{H}^2(\varphi_1(E)\setminus E) + h(\delta)\delta^2. \tag{4.4.19}$$

Since $E\supset L$, we have that $\varphi_1(E)\supset L_1$, and then we get that

$$\mathcal{H}^2(E\backslash\varphi_1(E)) = \mathcal{H}^2((E\backslash L_1)\backslash\varphi_1(E)) = \mathcal{H}^2\left((E\setminus L_1)\setminus\varphi(E\setminus L_1)\right). \tag{4.4.20}$$

We know that $F=\overline{E\setminus L_1}$ and $\mathcal{H}^2(F\cap L_1)=0$, thus

$$\mathcal{H}^2(F\setminus\varphi_1(F)) = \mathcal{H}^2(E\setminus\varphi_1(E)). \tag{4.4.21}$$

Similarly, we can get that

$$\mathcal{H}^2(\varphi_1(E)\setminus E) = \mathcal{H}^2(\varphi(E\setminus L_1)\setminus E) \leq \mathcal{H}^2(\varphi(E\setminus L_1)\setminus(E\setminus L_1)), \tag{4.4.22}$$

thus

$$\mathcal{H}^2(\varphi_1(E)\setminus E) \leq \mathcal{H}^2(\varphi_1(F)\setminus F). \tag{4.4.23}$$

From inequalities (4.4.19), (4.4.21) and (4.4.23), we obtain that

$$\mathcal{H}^2(F\setminus\varphi_1(F)) \leq \mathcal{H}^2(\varphi_1(F)\setminus F) + h(\delta)\delta^2.$$

Applying Proposition 20.9 in [11], we get that F is (U,h)-sliding-almost-minimal. □

If Ω, L_1, U, E and F are as in lemma 4.21, and we suppose that $0\in F$, then $\theta_F(0)$ can only take two values $\frac{1}{2}$ and $\frac{3}{4}$. Indeed, since $L_1\subset E$, any blow-up limit Z of E at 0 is a sliding minimal cone which contains the boundary L_1, thus $Z=L_1$ or $Z=L_1\cup Z'$, Z' is a sliding minimal cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$, hence the density $\theta_E(0)$ can only take three values, 1, $\frac{3}{2}$ and $\frac{7}{4}$. But if $\theta_E(0)=1$, then by Lemma 4.20, we can see that $0\notin F$. Therefore, $\theta_E(0)=\frac{3}{2}$ or $\frac{7}{4}$. We see that

$$\theta_E(0,r) = \theta_F(0,r)+1, \tag{4.4.24}$$

thus

$$\theta_E(x) = \theta_F(x)+1,$$

and $\theta_F(0)=\frac{1}{2}$ or $\frac{3}{4}$.

Lemma 4.22. *Let Ω and L_1 be as in (4.2.1), $\Pi_{L_1} : \mathbb{R}^3 \to L_1$ be the orthogonal projection onto L_1. Suppose that U is an open set, $E \subset \Omega$ is (U, h)-sliding-almost-minimal, $0 \in E \cap L_1 \cap U$ and $\theta_E(0) = \frac{1}{2}$ or $\frac{3}{4}$. If $\varepsilon > 0$ is small enough, $B(0, 2r) \subset U$, and for any $x \in E \cap L_1 \cap B(0, r)$, and any $0 < \rho \leq 3r/5$, there exists a sliding minimal cone Z_x^ρ of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ centered at x, such that*

$$d_{x,\rho}(E, Z_x^\rho) \leq \varepsilon,$$

then for any $z \in E \cap B(0, r/5)$, we can find a point $a \in E \cap L_1 \cap B(0, 3r/5)$ such that

$$|\Pi_{L_1}(z) - a| \leq 8\varepsilon\, |z - a|\,. \tag{4.4.25}$$

Proof. For any $z \in E \cap B(0, r/5)$, we put $z' = \Pi_{L_1}(z)$. We take a point $a \in E \cap L_1$ such that

$$|z' - a| \leq (1 + \varepsilon)\mathrm{dist}(z', E \cap L_1).$$

If $z' \in E \cap L_1$, $a = z' \in B(0, r/5)$, then nothing needs to be done. If $z' \notin E \cap L_1$, we claim that a is a point which we desire.

It is quite easy to see that $a \in B(0, 3r/5)$; otherwise

$$\frac{2r}{5} \leq |z' - a| \leq (1 + \varepsilon)\mathrm{dist}(z', E \cap L_1) \leq (1 + \varepsilon)|z' - 0| \leq (1 + \varepsilon)\frac{r}{5};$$

this gives a contradiction.

We put $\rho = 2|a - z|$. Since $d_{a,\rho}(E, Z_a^\rho) \leq \varepsilon$, and Z_a^ρ is perpendicular to L_1, we can find $z'' \in Z_a^\rho \cap L_1$ such that $|z' - z''| \leq \varepsilon\rho$.

We claim that $|z'' - a| \leq 3\varepsilon\rho$; once we have proved our claim, we can get that

$$|\Pi_{L_1}(z) - a| \leq |z' - z''| + |z'' - a| \leq 4\varepsilon\rho = 8\varepsilon|x - z|$$

We assume, for the sake of a contradiction, that $|z'' - a| > 3\varepsilon\rho$, then

$$|a - z'| \geq |a - z''| - |z' - z''| > 2\varepsilon\rho.$$

If $B(z'', 3\varepsilon\rho/2) \cap E \cap L_1 \neq \emptyset$, we take $x \in B(z'', 3\varepsilon\rho/2) \cap E \cap L_1$, then

$$|z' - x| \leq |z' - z''| + |z'' - x| \leq \frac{5}{2}\varepsilon\rho,$$

and

$$|z - x'| \geq \mathrm{dist}(z', E \cap L_1) \geq \frac{1}{1 + \varepsilon}|z' - a| \geq \frac{2\varepsilon\rho}{1 + \varepsilon},$$

thus

$$\frac{2\varepsilon\rho}{1 + \varepsilon} \leq \frac{5}{2}\varepsilon\rho;$$

this is a contradiction.

If $B(z'', 3\varepsilon\rho/2) \cap E \cap L_1 = \emptyset$, we can construct a projection to show that E is not almost minimal.

$\square$

Lemma 4.23. *Let Ω, L_1 be as in (4.2.1), U an open set with $0 \in U$. Suppose that $E \subset \Omega$ is (U,h)-sliding-almost-minimal. If $\theta_E(0) = \frac{1}{2}$, then for each small $\tau > 0$, we can find a radius $r > 0$, a sliding minimal cone Z of type $\mathbb{P}_+$ and a biHölder map $\phi : B(0,3r/2) \cap \Omega \to B(0,2r) \cap \Omega$ such that*

$$\begin{gathered}
\phi(x) \in L_1 \text{ for } x \in L_1 \cap B(0,3r/2), \ \|f - \mathrm{id}\|_\infty \le \tau r, \\
C|z-y|^{1+\tau} \le |\phi(z) - \phi(y)| \le C^{-1}|z-y|^{\frac{1}{1+\tau}}, \\
B(0,r) \cap \Omega \subset \phi\left(B\left(0, \frac{3r}{2}\right) \cap \Omega\right) \subset B(0,2r) \cap \Omega, \\
E \cap B(0,r) \subset \phi\left(Z \cap B\left(0, \frac{3r}{2}\right)\right) \subset E \cap B(0,2r),
\end{gathered} \tag{4.4.26}$$

where C is a constant which only depends on τ and r.

Proof. We can assume that U is an open ball $B(0,R)$ for some $R > 0$. For any $\tau \in (0,1]$, we let $\varepsilon(\tau)$ be as in Lemma 4.19, we suppose that τ is so small, that

$$(1+\varepsilon(\tau))e^{(\lambda+\alpha)\varepsilon(\tau)} < \frac{3}{2},$$

where λ is taken as in Proposition 5.24 in [8], and α is taken as in Theorem 28.7 in [11]. Let $\tau_2 > 0$ and $\varepsilon(\tau_2)$ be as in Lemma 4.19 and such that $100\tau_2 \le \tau$ and

$$\left(\frac{1}{2} + \varepsilon(\tau_2)\right) e^{\alpha\varepsilon(\tau)} < \frac{3}{4}.$$

We take $0 < \tau_1 \le \min\{\tau_2, \varepsilon(\tau_2)\}/100$, and let τ_1, $\varepsilon(\tau_1)$ be also as in Lemma 4.19. We always suppose that $\varepsilon(\tau_1) < \varepsilon(\tau_2) < \varepsilon(\tau)$.

By Theorem 28.7 in [11], we can find $r_0 \in (0,R)$ such that

$$h(2r_0) \le \varepsilon(\tau_1), \ A(r_0) \le \varepsilon(\tau_1), \ \theta_E(0,r_0) \le \frac{1}{2} + \varepsilon(\tau_1),$$

where

$$A(r) = \int_0^{2r} \frac{h(t)}{t} dt.$$

By using Lemma 4.19, for any $r \in (0, 9r_0/10]$, there exists a minimal cone Z^r of type $\mathbb{P}_+$ center at 0 such that

$$d_{0,r}(E, Z^r) \le \tau_1 \tag{4.4.27}$$

and for any ball $B(y,t) \subset B(0,r)$,

$$\left|\mathcal{H}^2(Z^r \cap B(y,t)) - \mathcal{H}^2(E \cap B(y,t))\right| \leq \tau_1 r^2. \tag{4.4.28}$$

First, we consider any point $x \in E \cap L_1 \cap B(0, r_0/2)$. By (4.4.28), if we take $r = 9r_0/10$, we will get that

$$\begin{aligned}\mathcal{H}^2(E \cap B(x,t)) &\leq \mathcal{H}^2\left(Z^{9r_0/10} \cap B(x,t)\right) + \tau_1\left(\frac{9r_0}{10}\right)^2\\ &\leq \frac{\pi}{2}t^2 + \tau_1\left(\frac{9r_0}{10}\right)^2\end{aligned}$$

from this inequality, by taking $t = r_0/3$, we can get that

$$\theta_E\left(x, \frac{r_0}{3}\right) < \frac{1}{2} + 30\tau_1 < \frac{1}{2} + \varepsilon(\tau_2). \tag{4.4.29}$$

By using Theorem 28.7 in [11], we get that

$$\theta_E(x) \leq \theta_E\left(x, \frac{r_0}{3}\right) e^{\alpha A(r_0/3)} < \frac{3}{4},$$

thus

$$\theta_E(x) = \frac{1}{2}. \tag{4.4.30}$$

By Lemma 4.19, we can find minimal sliding cone Z_x^ρ for any $0 < \rho \leq 3r_0/10$ such that

$$d_{x,\rho}(E, Z_x^\rho) \leq \tau_2 \tag{4.4.31}$$

and for any ball $B(y,t) \subset B(x,\rho)$,

$$\left|\mathcal{H}^2(E \cap B(y,t)) - \mathcal{H}^2(Z_x^\rho \cap B(y,t))\right| \leq \tau_2\rho^2. \tag{4.4.32}$$

We now consider any point $z \in E \cap B(0, r_0/10) \setminus L_1$. From Lemma 4.22, we get that

$$|\Pi_{L_1}(z) - x| \leq 8\tau_2 |z - x|,$$

thus

$$\operatorname{dist}(z, L_1) = |z - \Pi_{L_1}(z)| \geq |z - x| - |\Pi_{L_1}(z) - x| \geq (1 - 8\tau_2)|z - x|. \tag{4.4.33}$$

We take $r_1 = \frac{1}{2}\operatorname{dist}(z, L_1)$, and $\rho = |z - x| + r_1$, then $\rho < \frac{3r_0}{10}$. We take Z_x^ρ as in (4.4.31), then $B(z, r_1) \subset B(x, \rho)$, thus

$$\mathcal{H}^2(E \cap B(z, r_1)) \leq \mathcal{H}^2(Z_x^\rho \cap B(z,r)) + \tau_2\rho^2 \leq \pi r_1^2 + \tau_2\rho^2,$$

hence

$$\theta_E(z, r_1) = \frac{\mathcal{H}^2(E \cap B(z, r_1))}{\pi r_1^2} \leq 1 + \frac{\tau_2 \rho^2}{\pi r_1^2} < 1 + 10\tau_2 < 1 + \varepsilon(\tau), \quad (4.4.34)$$

but we know that

$$\theta_E(z) \geq 1,$$

hence by using a monotonicity property, Proposition 5.24 in [8], we have that

$$\theta_E(z) \leq \theta_E(z, r_1) \exp(\lambda A(r_1)) < \frac{3}{2},$$

thus

$$\theta_E(z) = 1. \quad (4.4.35)$$

For any $r \in (0, \frac{9}{10} r_1]$, we can apply Lemma 16.11 in [8], there exists a plane $Z(z, r)$ through z such that

$$d_{z,r}(E, Z(z, r)) \leq \tau. \quad (4.4.36)$$

For any $r \in (\frac{9}{10} r_1, \frac{1}{5} r_0]$, we put $\rho_r = |z - x| + r$, then $\rho_r \leq \frac{3}{10} r_0$, and $B(z, r) \subset B(x, \rho_r)$. We take $Z_x^{\rho_r}$ as in (4.4.31), then

$$d_{z,r}(E, Z_x^{\rho_r}) \leq \frac{\rho_r}{r} d_{x,\rho_r}(E, Z(x, \rho_r)) \leq \frac{\rho_r}{r} \tau_2 \leq 5\tau_2. \quad (4.4.37)$$

We do not know whether or not the sliding minimal cone $Z_x^{\rho_r}$ passes through the point z, but we can do a translation of $Z_x^{\rho_r}$ such that it is centered at $\Pi_{L_1}(z)$, we denote it by $Z(z, r)$, i.e. $Z(z, r) = Z_x^{\rho_r} - (x - \Pi_{L_1}(z))$. Then $Z(z, r)$ is a sliding minimal cone contains z, and

$$d_{z,r}(E, Z(z, r)) \leq \frac{|x - \Pi_{L_1}(z)|}{r} + d_{z,r}(E, Z_x^{\rho_r}) < 20\tau_2 < \tau. \quad (4.4.38)$$

It follows from (4.4.36) and (4.4.38) that, for any $z \in E \cap B(0, r_0/10) \setminus L_1$, for any $r \in (0, r_0/5]$, there exist a cone $Z(z, r)$ such that

$$d_{z,r}(E, Z(z, r)) \leq \tau, \quad (4.4.39)$$

where $Z(z, r)$ is a plane when r is small, $Z(x, r)$ is a half plane when r is large.

From the inequalities (4.4.32) and (4.4.39), we get that, for any $x \in E \cap B(0, r_0/10)$ and any $r \in (0, r_0/5]$, we can find a cone $Z(x, r)$ though x such that

$$d_{x,r}(E, Z(x, r)) \leq \tau.$$

where $Z(x,r)$ is a minimal cone when $0 < r < \operatorname{dist}(x, L_1)$, and $Z(x,r)$ is a sliding minimal cone of type $\mathbb{P}_+$ when $\operatorname{dist}(x, L_1) \le r \le r_0/5$.

By Corollary 4.18, we can find a biHölder map $\phi : B(0, 3r_0/20) \cap \Omega \to B(0, r_0/5) \cap \Omega$ and a sliding minimal cone Z_0 of type $\mathbb{P}_+$ such that (4.4.26) holds with $r = r_0/10$. $\square$

Lemma 4.24. *Let Ω, L_1 be as in (4.2.1), U an open set with $0 \in U$. Suppose that $E \subset \Omega$ is (U,h)-sliding-almost-minimal. If $E \supset L_1$ and $\theta_E(0) = \frac{3}{2}$, then for each small $\tau > 0$, we can find a radius $r > 0$, a biHölder map $\phi : B(0, 3r/2) \cap \Omega \to B(0, 2r) \cap \Omega$ and a sliding minimal cone Z of type $\mathbb{P}_+$ such that*

$$\begin{aligned} &\phi(x) \in L_1 \text{ for } x \in L_1 \cap B(0, 3r/2),\ \|f - \mathrm{id}\|_\infty \le \tau r, \\ &C\,|z-y|^{1+\tau} \le |\phi(z) - \phi(y)| \le C^{-1}\,|z-y|^{\frac{1}{1+\tau}}, \\ &B(0,r) \cap \Omega \subset \phi\left(B\left(0, \frac{3r}{2}\right) \cap \Omega\right) \subset B(0,2r) \cap \Omega, \\ &E \cap B(0,r) \subset \phi\left((Z \cup L_1) \cap B\left(0, \frac{3r}{2}\right)\right) \subset E \cap B(0,2r), \end{aligned} \tag{4.4.40}$$

where C is a constant which only depends on τ and r.

Proof. We put $F = \overline{E \setminus L_1}$, then F is also (U,δ,h)-sliding-almost-minimal. By lemma 4.23, for each small $\tau > 0$, we can find $r > 0$, a biHölder map $\phi : B(0, 3r/2) \cap \Omega \to B(0, 2r) \cap \Omega$ and a sliding minimal cone Z of type $\mathbb{P}_+$ such that

$$\begin{aligned} &\phi(x) \in L_1 \text{ for } x \in L_1 \cap B(0, 3r/2),\ \|f - \mathrm{id}\|_\infty \le \tau r, \\ &C\,|x-y|^{1+\tau} \le |\phi(x) - \phi(y)| \le C^{-1}\,|x-y|^{\frac{1}{1+\tau}}, \\ &B(0,r) \cap \Omega \subset \phi\left(B\left(0, \frac{3r}{2}\right) \cap \Omega\right) \subset B(0,2r) \cap \Omega, \\ &F \cap B(0,r) \subset \phi\left(Z \cap B\left(0, \frac{3r}{2}\right)\right) \subset F \cap B(0,2r). \end{aligned} \tag{4.4.41}$$

Thus

$$E \cap B(0,r) \subset \phi\left((Z \cup L_1) \cap B\left(0, \frac{3r}{2}\right)\right) \subset E \cap B(0,2r).$$

$\square$

Remark 4.25. *Suppose that Ω, L_1 and U are as in Lemma 4.23, and that $E \subset \Omega$ is a (U,h)-sliding-almost-minimal set. Suppose that $\theta_E(0) = \frac{1}{2}$, or* *that* $\theta_E = \frac{3}{2}$ *and* $L_1 \subset E$. *If* $\tau \in (0,1)$ *is small enough, we can find* $\varepsilon'(\tau) > 0$ *such that when the radius* $r > 0$ *is such that*

$$B(0,10r) \subset U, h(20r) \le \varepsilon'(\tau), \int_0^{20r} \frac{h(t)}{t} dt \le \varepsilon'(\tau), \theta_E(0,10r) \le \theta_E(0) + \varepsilon'(\tau)$$

then for any $x \in E \cap B(0,r)$ *and any* $0 < t \le 2r$, *we can find a cone or sliding minimal cone* $Z(x,t)$ *that depends on* t *such that*

$$d_{x,t}(E, Z(x,t)) \le \tau,$$

where $Z(x,t)$ *is a minimal cone when* $0 < t < \mathrm{dist}(x, L_1)$, *and* $Z(x,t)$ *is a sliding minimal cone when* $\mathrm{dist}(x, L_1) \le t \le 2r$.

Indeed, when we look at the proof of Lemma 4.23, we let $\tau \in (0,1)$ be such that

$$\left(\frac{1}{2} + \varepsilon(\tau)\right) e^{(\lambda+\alpha)\varepsilon(\tau)} < \frac{3}{4}.$$

Then we take

$$\tau_1 = \min\left\{\frac{\tau}{10^4}, \frac{1}{100}\varepsilon\left(\frac{\tau}{100}\right)\right\},$$

and let $\varepsilon(\tau_1)$ be as in Lemma 4.19. Finally, $\varepsilon'(\tau) = \varepsilon(\tau_1)$ will be what we desire.

Lemma 4.26. *Let* Ω, L_1 *be as in* (4.2.1), U *an open set with* $0 \in U$. *Suppose that* $F \subset \Omega$ *is an* (U,h)*-sliding-almost-minimal set. If* $\theta_F(0) = \frac{3}{4}$, *then for each small* $\tau > 0$, *we can find a radius* $r > 0$, *a biHölder map* $\phi : B(0,3r/2) \cap \Omega \to B(0,2r) \cap \Omega$ *and a sliding minimal cone of type* $\mathbb{Y}_+$ *such that*

$$\begin{aligned} &\phi(x) \in L_1 \text{ for } x \in L_1 \cap B()0,3r/2, \ \|\phi - \mathrm{id}\|_\infty \le \tau r, \\ &C\,|z-y|^{1+\tau} \le |\phi(z) - \phi(y)| \le C^{-1}\,|z-y|^{\frac{1}{1+\tau}}, \\ &B(0,r) \cap \Omega \subset \phi\left(B\left(0, \frac{3r}{2}\right) \cap \Omega\right) \subset B(0,2r) \cap \Omega, \\ &F \cap B(0,r) \subset \phi\left(Z \cap B\left(0, \frac{3r}{2}\right)\right) \subset F \cap B(0,2r), \end{aligned} \tag{4.4.42}$$

where C *is a constant which only depends on* τ *and* r.

Proof. As in Lemma 4.23, we can assume U is an open ball $B(0,R)$ for some $R > 0$. Let $\tau > 0$ be a positive number, and let $\varepsilon(\tau)$ be as in Lemma 4.19; we suppose τ small enough so that

$$(1 + \varepsilon(\tau))e^{(\lambda+\alpha)\varepsilon(\tau)} < \frac{3}{2},$$

where λ is taken as in Proposition 5.24 in [8], and α is taken as in Theorem 28.7 in [11]. Let $\tau_2 > 0$ and $\varepsilon(\tau_2)$ be as in Lemma 4.19 so that $100\tau_2 \leq \tau$ and

$$\left(\frac{1}{2} + \varepsilon(\tau_2)\right) e^{\alpha\varepsilon(\tau)} < \frac{3}{4}, \ \left(\frac{3}{2} + \varepsilon(\tau_2)\right) e^{\lambda\varepsilon(\tau)} < d_T,$$

where d_T is the constant which is considered in Lemma 14.12 in [8]. We take $0 < \tau_1 \leq \min\{\tau_2, \varepsilon(\tau_2)\}/100$. Let τ_1 and $\varepsilon(\tau_1)$ be as in Lemma 4.19. We suppose that $\varepsilon(\tau_1) < \varepsilon(\tau_2) < \varepsilon(\tau)$.

By Theorem 28.7 in [11], there exist $0 < r_0 < R$ such that

$$h(2r_0) \leq \varepsilon(\tau_1), \ A(r_0) < \varepsilon(\tau_1)$$

and

$$\theta_F(0, r_0) \leq \frac{3}{4} + \varepsilon(\tau_1),$$

where

$$A(r) = \int_0^{2r} \frac{h(t)}{t} dt.$$

By using Lemma 4.19, for any $\rho \in (0, 9r_0/10]$ there exists a minimal cone Z^ρ of type $\mathbb{Y}_+$ center at 0 such that

$$\begin{gathered} d_{0,\rho}(F, Z^\rho) \leq \tau_1, \\ \left|\mathcal{H}^2(Z^\rho \cap B(y,t)) - \mathcal{H}^2(E \cap B(y,t))\right| \leq \tau_1 \rho^2, \\ \text{for any ball } B(y,t) \subset B(0,\rho). \end{gathered} \tag{4.4.43}$$

First, for any $x \in F \cap L_1 \cap B(0, r_0/2) \setminus \{0\}$, we take $\rho = 2|x|$ and $t = |x|$, then by (4.4.43), we have

$$\begin{aligned} \mathcal{H}^2(F \cap B(x,t)) &\leq \mathcal{H}^2\left(Z^\rho \cap B(x,t)\right) + \tau_1\rho^2 \\ &\leq \frac{\pi}{2} t^2 + \tau_1 \rho^2 \end{aligned}$$

from this inequality, we can get that

$$\theta_F\left(x, |x|\right) = \frac{\mathcal{H}^2(E \cap B(x,|x|))}{\pi |x|^2} \leq \frac{1}{2} + 4\tau_1 < \frac{1}{2} + \varepsilon(\tau_2) \tag{4.4.44}$$

Applying Theorem 28.7 in [11], we get that

$$\theta_F(x) \leq \theta_F(x, |x|) e^{\alpha A(|x|)} < \frac{3}{4},$$

thus

$$\theta_F(x) = \frac{1}{2}.$$

Taking $r_1 = |x|$, we get from (4.4.44) that

$$\theta_F(x, r_1) \leq \theta_F(x) + \varepsilon(\tau_2).$$

By Lemma 4.19, for any $0 < \rho \leq 9r_1/10$, there exists a sliding minimal cone Z_x^ρ centered at x of type $\mathbb{P}_+$ such that

$$d_{x,r}(F, Z_x^\rho) \leq \tau_2, \tag{4.4.45}$$

and for any ball $B(y,t) \subset B(x,r)$,

$$\left|\mathcal{H}^2(F \cap B(y,t)) - \mathcal{H}^2(Z_x^\rho \cap B(y,t))\right| \leq \tau_2 \rho^2. \tag{4.4.46}$$

For $9r_1/10 \leq \rho \leq 3r_0/10$, we have that

$$d_{x,\rho}(F, Z^{r_1+\rho}) \leq \frac{r_1+\rho}{\rho} d_{0,r_1+\rho}(F, Z^{r_1+\rho}) \leq \frac{19}{9}\tau_1.$$

Since $d_{0,r_1+\rho}(F, Z^{r_1+\rho}) \leq \tau_1$, there exists a point $x' \in Z^{r_1+\rho} \cap L_1$ such that $|x - x'| \leq (r_1+\rho)\tau_1$. We take $Z(x,\rho) = Z^{r_1+\rho} + x - x'$, that is a translation of $Z^{r_1+\rho}$; it is a siliding minimal through the point x, and

$$d_{x,\rho}(F, Z(x,\rho)) \leq \frac{|x-x'|}{\rho} + d_{x,\rho}(F, Z^{r_1+\rho}) < 5\tau_1 < \tau_2. \tag{4.4.47}$$

It follows from (4.4.45) and (4.4.47) that, for any $x \in F \cap L_1 \cap B(0, r_0/2)$, and any $0 < \rho < 3r_0/10$, there exists a sliding minimal cone $Z(x,\rho)$ centered at x, either of type $\mathbb{P}_+$ or of type $\mathbb{Y}_+$, such that

$$d_{x,\rho}(F, Z(x,\rho)) \leq \tau_2. \tag{4.4.48}$$

Next, We consider $z \in (F \setminus L_1) \cap B(0, r_0/10)$. If $\mathrm{dist}(z, L_1) < \frac{1}{3}|z|$, we take a point $a \in F \cap L_1 \cap B(0, \frac{r_0}{5})$ such that

$$|\Pi_{L_1}(z) - a| \leq 8\tau_2 |z - a|.$$

We take $r_2 = \frac{1}{2}\mathrm{dist}(x, L_1)$ and $\rho = |z-a| + r_2$, then

$$|z - a| \leq |z - \Pi_{L_1}(z)| + |\Pi_{L_1}(z) - a| \leq 2r_2 + 8\tau_2 |z-a|,$$

thus

$$|z - a| \leq \frac{2}{1-8\tau_2} r_2,$$

and

$$\rho \leq \left(1 + \frac{2}{1-8\tau_2}\right) r_2 \leq \frac{75}{23} r_2.$$

Since $2r_2 = \mathrm{dist}(z, L_1) \leq \frac{1}{3}|z|$, we have that

$$|\Pi_{L_1}(z)| \geq 2\sqrt{2}\,|z - \Pi_{L_1}(z)| \geq 4\sqrt{2}r_2,$$

thus

$$|a| \geq |\Pi_{L_1}(z)| - |\Pi_{L_1}(z) - a| \geq 4\sqrt{2}r_2 - \frac{16\tau_2}{1-8\tau_2}r_2 > 4r_2.$$

Hence

$$\rho \leq \frac{9}{10}\,|a|\,.$$

Consider the sliding minimal cone Z_a^ρ as in (4.4.45); it is a minimal cone centered at point a of type $\mathbb{P}_+$. Since $B(z, r_2) \subset B(a, \rho)$, we deduce from (4.4.46) that

$$\begin{aligned} \mathcal{H}^2\left(F \cap B(z, r_2)\right) &\leq \mathcal{H}^2\left(Z_a^\rho \cap B(z, r_2)\right) + \tau_2\rho^2 \\ &\leq \pi r_2^2 + \tau_2\rho^2, \end{aligned}$$

thus

$$\theta_F\left(z, r_2\right) \leq 1 + \left(\frac{73}{23}\right)^2 \frac{\tau_2}{\pi} \leq 1 + 4\tau_2 \leq 1 + \varepsilon(\tau). \tag{4.4.49}$$

Applying Proposition 5.24 in [8], we can get that

$$\theta_F(z) \leq \theta_F(z, r_2)e^{\lambda A(r_2)} < \frac{3}{2},$$

thus

$$\theta_F(z) = 1.$$

By Lemma 16.11 in [8], for any $\rho \in (0, 9r_2/10]$, there exist a plane $Z(z, \rho)$ through z such that

$$d_{z,\rho}(F, Z(z,\rho)) \leq \tau. \tag{4.4.50}$$

For $\rho \in (9r_2/10, r_0/10]$, we put $r_\rho = |z - a| + \rho$, then

$$r_\rho \leq \frac{r_0}{5} + \frac{r_0}{10} \leq \frac{3r_0}{10}.$$

Consider the sliding minimal cone $Z(a, r_\rho)$ as in (4.4.48), we can get that

$$d_{z,\rho}(F, Z(a, r_\rho) \leq \frac{r_\rho}{\rho} d_{a,r}(F, Z(a, r_\rho)) \leq \left(1 + \frac{|z-a|}{\rho}\right)\tau_2 \leq \frac{7}{2}\tau_2.$$

We now take

$$Z(z, \rho) = Z(a, r_\rho) + \Pi_{L_1}(z) - a.$$

It is a sliding minimal cone centered at $\Pi_{L_1}(z)$, thus through the point z, which satisfy that

$$d_{z,\rho}(F, Z(z,\rho)) \leq \frac{|\Pi_{L_1}(z) - a|}{\rho} + d_{z,\rho}(F, Z(a, r_\rho)) < 4\tau_2. \tag{4.4.51}$$

It follows from (4.4.50) and (4.4.51) that, in the case $z \in B(0, r_0/10) \cap F \setminus L_1$ and $\mathrm{dist}(z) < |z|/3$, for $0 < \rho \leq \frac{r_0}{10}$, we can find a cone $Z(z,\rho)$ such that

$$d_{z,\rho}(F, Z(z,\rho)) \leq \tau, \tag{4.4.52}$$

where $Z(z,\rho)$ is a minimal cone when ρ is small, and $Z(z,\rho)$ is a sliding minimal cone when ρ is large.

We now consider the case when $z \in (F \setminus L_1) \cap B(0, r_0/10)$ with $\mathrm{dist}(z, L_1) \geq \frac{1}{3}|z|$. We take $r_3 = \mathrm{dist}(z, L_1)$, and put $\rho_3 = |z| + r_3$, then $\rho_3 \leq 4r_3 < \frac{4r_0}{10}$. We take Z^{ρ_3} a minimal cone as in (4.4.43), then we can get that

$$\mathcal{H}^2(F \cap B(z, r_3)) \leq \mathcal{H}^2(Z^{\rho_3} \cap B(z, r_3)) + \tau_1 \rho_3^2 \leq \frac{3}{2}\pi r_3^2 + \tau_1 \rho_3^2,$$

thus

$$\theta_F(z, r_3) \leq \frac{3}{2} + \frac{16}{\pi}\tau_1 < \frac{3}{2} + \varepsilon(\tau_2). \tag{4.4.53}$$

Applying Proposition 5.24 in [8], we get that

$$\theta_F(z) \leq \theta_F(z, r_3) e^{\lambda A(r_3)} < d_T,$$

thus $\theta_F(z) = 1$ or $\theta_F(z) = \frac{3}{2}$.

Case 1. If $\theta(z) = \frac{3}{2}$, then for any $0 < \rho \leq \frac{9}{10} r_3$, by using Lemma 16.11 in [8], there exists a minimal cone $Z(z,\rho)$ centered at z of type $\mathbb{Y}$ such that

$$d_{x,\rho}(E, Z(x,\rho)) \leq \tau_2 \tag{4.4.54}$$

and for any ball $B(y,t) \subset B(x,\rho)$

$$\left|\mathcal{H}^2(E \cap B(y,t)) - \mathcal{H}^2(Z(x,\rho) \cap B(y,t))\right| \leq \tau_2 r_3^2. \tag{4.4.55}$$

For any $\rho \in (\frac{9}{10} r_3, \frac{4r_0}{5}]$, we put $r_\rho = |z| + \rho$, then $r_\rho \leq \frac{9r_0}{10}$, and $|z| \leq 3\mathrm{dist}(z, L_1) = 6r_3$, thus $r_\rho < 8r\rho$. Let Z^{r_ρ} be the sliding minimal cone which is considered in (4.4.43), then we can get that

$$d_{z,\rho}(F, Z^{r_\rho}) \leq \frac{r_\rho}{\rho} d_{0,r_\rho}(F, Z^{r_\rho}) \leq \frac{r_\rho}{\rho}\tau_1.$$

We take a point $z' \in Z^{r_\rho}$ such that $|z - z'| \leq r_\rho \tau_1$, and take $Z(z, \rho) = Z^{r_\rho} + \Pi_{L_1}(z) - \Pi_{L_1}(z')$, which through point z, we obtain that

$$d_{z,\rho}(F, Z(z,\rho)) \leq \frac{|\Pi_{L_1}(z) - \Pi_{L_1}(z')|}{\rho} + d_{z,\rho}(F, Z^{r_\rho}) \leq \frac{2r_\rho}{\rho}\tau_1 \leq 16\tau_1. \quad (4.4.56)$$

It follows from (4.4.54) and (4.4.56) that, when $z \in B(0, r_0/10) \cap F \setminus L_1$ and $\mathrm{dist}(z, L_1) \geq |z|/3$ with $\theta_F(z) = \frac{3}{2}$, for any $\rho \in (0, 4r_0/5]$, we can find a cone $Z(z,\rho)$ such that

$$d_{z,\rho}(F, Z(z,\rho)) \leq \tau_2, \quad (4.4.57)$$

where $Z(z,\rho)$ is a minimal cone when ρ small, and $Z(z,\rho)$ is a sliding minimal cone with boundary L_1 when ρ large.

Case 2. If $\theta_F(z) = 1$. We set

$$E_Y = \{0\} \cup \left\{ x \in F \;\middle|\; \theta(x) = \frac{3}{2} \right\},$$

and denote $\ell_Y(z) = \mathrm{dist}(z, E_Y)$. Using the same argument as in Lemma 16.25 in [8, in page 205], we get that for $\rho \in (0, \ell_Y(z)/3]$, there is a plane $Z(z,\rho)$ through x such that

$$d_{z,\rho}(F, Z(z,\rho)) \leq \tau. \quad (4.4.58)$$

For $\rho \in (\ell_Y(z)/3, r_0/10]$, we take a point $x \in E_Y$ such that

$$|z - x| \leq \frac{11}{10}\ell_Y(z),$$

and consider the cone $Z(x, r_\rho)$ as in (4.4.57), where $r_\rho = |z-x| + \rho$. We can get that

$$d_{z,\rho}(F, Z(x, r_\rho)) \leq \frac{r_\rho}{\rho} d_{x,r_\rho}(F, Z(x,r_\rho)) \leq \frac{r_\rho}{\rho}\tau_2.$$

By a similar argument as before, we can find a cone $Z(z,\rho)$ which is a translation of $Z(x, r_\rho)$ such that

$$d_{z,\rho}(F, Z(z,\rho)) \leq \frac{2r_\rho}{\rho}\tau_2 < 10\tau_2. \quad (4.4.59)$$

It follows that, when $z \in B(0, r_0/10) \cap F \setminus L_1$ and $\mathrm{dist}(z, L_1) \geq |z|/3$ with $\theta_F(z) = 1$, for any $\rho \in (0, r_0/10]$, we can find a cone $Z(z,\rho)$ such that

$$d_{z,\rho}(F, Z(z,\rho)) \leq \tau, \quad (4.4.60)$$

where $Z(z,\rho)$ is a minimal cone when ρ is small, and $Z(z,\rho)$ is a sliding minimal cone when ρ is large.

From inequalyties (4.4.48), (4.4.52), (4.4.57) and (4.4.60), we can say that for any $z \in B(0, r_0/10)$, and for any $\rho \in (0, r_0/10]$, there is a cone $Z(z, \rho)$ such that

$$d_{z,\rho}(F, Z(z, \rho)) \le \tau,$$

where $Z(z, \rho)$ is a minimal cone when $\rho < \mathrm{dist}(z, L_1)$, and $Z(z, \rho)$ is a sliding minimal cone when $\rho \ge \mathrm{dist}(z, L_1)$.

By Corollary 4.18, we can get our desired result. □

Lemma 4.27. *Let Ω, L_1 be as in (4.2.1), U an open set with $0 \in U$. Suppose that $E \subset \Omega$ is an (U, h)-sliding-almost-minimal set. If $\theta_E(0) = \frac{7}{4}$ and $E \supset L_1$, then for each small $\tau > 0$, we can find a radius $r > 0$, a biHölder map $\phi : B(0, 3r/2) \cap \Omega \to B(0, 2r) \cap \Omega$ and a sliding minimal cone Z of type $\mathbb{Y}_+$ such that*

$$\begin{gathered}
\phi(x) \in L_1 \text{ for } x \in L_1, \|f - \mathrm{id}\|_\infty \le \tau, \\
C\, |z - y|^{1+\tau} \le |\phi(z) - \phi(y)| \le C^{-1}\, |z - y|^{\frac{1}{1+\tau}}, \\
B(0, r) \cap \Omega \subset \phi\left(B\left(0, \frac{3r}{2}\right) \cap \Omega\right) \subset B(0, 2r) \cap \Omega, \\
E \cap B(0, r) \subset \phi\left((Z \cup L_1) \cap B\left(0, \frac{3r}{2}\right)\right) \subset E \cap B(0, 2r),
\end{gathered} \tag{4.4.61}$$

where C is a constant which only depends on τ and r.

Proof. We put $F = \overline{E \setminus L_1}$, then F is also (U, δ, h)-sliding-almost-minimal. By lemma 4.26, for each small $\tau > 0$, we can find $r > 0$, a biHölder map $\phi : B(0, 3r/2) \cap \Omega \to B(0, 2r) \cap \Omega$ and a sliding minimal cone Z of type $\mathbb{Y}_+$ such that

$$\begin{gathered}
\phi(x) \in L_1 \text{ for } x \in L_1 \cap B(0, 3r/2),\ \|f - \mathrm{id}\|_\infty \le \tau r, \\
C\, |x - y|^{1+\tau} \le |\phi(x) - \phi(y)| \le C^{-1}\, |x - y|^{\frac{1}{1+\tau}}, \\
B(0, r) \cap \Omega \subset \phi\left(B\left(0, \frac{3r}{2}\right) \cap \Omega\right) \subset B(0, 2r) \cap \Omega, \\
F \cap B(0, r) \subset \phi\left(Z \cap B\left(0, \frac{3r}{2}\right)\right) \subset F \cap B(0, 2r).
\end{gathered} \tag{4.4.62}$$

Thus

$$E \cap B(0, r) \subset \phi\left((Z \cup L_1) \cap B\left(0, \frac{3r}{2}\right)\right) \subset E \cap B(0, 2r).$$

□

Remark 4.28. *Suppose that Ω, L_1 and U are as in Lemma 4.26, and that $E \subset \Omega$ is a (U,h)-sliding-almost-minimal set. Suppose that $\theta_E(0) = \frac{3}{4}$, or that $\frac{7}{4}$ and $E \supset L_1$. If $\tau \in (0,1)$ is small enough, we can find $\varepsilon'(\tau) > 0$ such that when $r > 0$ is such that*

$$B(0,10r) \subset U, h(20r) \leq \varepsilon'(\tau), \int_0^{20r} \frac{h(t)}{t} dt \leq \varepsilon'(\tau), \theta_E(0,10r) \leq \theta_E(0) + \varepsilon'(\tau),$$

then for any $x \in E \cap B(0,r)$ and any $0 < t \leq 2r$, we can find a minimal cone or sliding minimal cone $Z(x,t)$ such that

$$d_{x,t}(E, Z(x,t)) \leq \tau,$$

where $Z(x,t)$ is a minimal cone when $0 < t < \operatorname{dist}(x, L_1)$, and $Z(x,t)$ is a sliding minimal cone when $\operatorname{dist}(x, L_1) \leq t \leq 2r$.

Indeed, we can take $\tau \in (0,1)$ be such that

$$\left(\frac{1}{2} + \varepsilon(\tau)\right) e^{\alpha\varepsilon(\tau)} < \frac{3}{4} \text{ and } \left(\frac{3}{2} + \varepsilon(\tau) e^{\lambda\varepsilon(\tau)}\right) < d_T,$$

then we take

$$\tau_1 = \min\left\{\frac{\tau}{10^4}, \frac{1}{100}\varepsilon\left(\frac{\tau}{100}\right)\right\},$$

and let $\varepsilon(\tau_1)$ be as in Lemma 4.19. We can check from the proof of Lemma 4.26 that $\varepsilon'(\tau) = \varepsilon(\tau_1)$ is the number what we desire.

Proposition 4.29. *Let Ω, L_1 be as in (4.2.1), U an open set. Let $E \subset \Omega$ be an (U,h)-sliding-almost-minimal set, and $x \in L_1 \cap U$ be a point. We suppose that $\theta_E(x) \in \{1/2, 3/4\}$, or that $\theta_E \in \{3/2, 7/4\}$ and $E \supset L_1$, then for each small $\tau > 0$, we can find a radius $r > 0$, a sliding minimal cone Z centered at x and a biHölder map $\phi : B(x, 3r/2) \cap \Omega \to B(x, 2r) \cap \Omega$ such that*

$$\begin{gathered}
\phi(z) \in L_1 \text{ for } z \in L_1 \cap B(x, 3r/2),\ \|f - \mathrm{id}\|_\infty \leq \tau r,\\
C|z-y|^{1+\tau} \leq |\phi(z) - \phi(y)| \leq C^{-1}|z-y|^{\frac{1}{1+\tau}},\\
B(0,r) \cap \Omega \subset \phi\left(B\left(0, \frac{3r}{2}\right) \cap \Omega\right) \subset B(0,2r) \cap \Omega,\\
E \cap B(0,r) \subset \phi\left(Z \cap B\left(0, \frac{3r}{2}\right)\right) \subset E \cap B(0,2r).
\end{gathered} \tag{4.4.63}$$

In addition, if $\theta_E(x) = \frac{1}{2}$, Z is a cone of type $\mathbb{P}_+$; if $\theta_E(x) = \frac{3}{4}$, Z is a cone of type $\mathbb{Y}_+$; if $\theta_E(x) = \frac{3}{2}$, $Z = Z' \cup L_1$ where Z' is a cone of type $\mathbb{P}_+$; if $\theta_E(x) = \frac{7}{4}$, $Z = Z' \cup L_1$ where Z' is a cone of type $\mathbb{Y}_+$.

The proof immediately follows from Lemma 4.23, Lemma 4.24, Lemma 4.26 and Lemma 4.27.

Corollary 4.30. *Let Ω, L_1 be as in (4.2.1), U an open set. Let $E \subset \Omega$ be an (U, h)-sliding-almost-minimal set with $E \supset L_1$. Then for each small $\tau > 0$ and each $x \in L_1 \cap U$, we can find a radius $r > 0$, a sliding minimal cone Z and a biHölder map $\phi : B(x, 3r/2) \to B(x, 2r)$ such that*

$$\begin{aligned}
&\phi(x) \in L_1 \text{ for } x \in L_1 \cap B(x, 3r/2),\ \|f - \mathrm{id}\|_\infty \leq \tau r,\\
&C|x-y|^{1+\tau} \leq |\phi(x) - \phi(y)| \leq C^{-1}|x-y|^{\frac{1}{1+\tau}},\\
&B(0, r) \cap \Omega \subset \phi\left(B\left(0, \frac{3r}{2}\right) \cap \Omega\right) \subset B(0, 2r) \cap \Omega,\\
&E \cap B(0, r) \subset \phi\left(Z \cap B\left(0, \frac{3r}{2}\right)\right) \subset E \cap B(0, 2r),
\end{aligned} \tag{4.4.64}$$

where C is a constant which only depends on τ and r.

Proof. Since $E \supset L_1$, any blow-up limit F of E at x contains L_1, so it is a sliding minimal cone contains L_1. By Theorem 4.15, we can get that $F = L_1$ or $F = Z \cup L_1$, where Z is a cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$. If $F = L_1$, by Lemma 4.20, then there exists a ball $B(x, r)$ such that $E \cap B(x, r) = L_1 \cap B(x, r)$, thus (4.4.64) hold automatically. If $F \neq L_1$, then $F = Z \cup L_1$ where Z is a sliding minimal cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$; we get that $\theta_E(x) = \frac{3}{2}$ or $\frac{7}{4}$, by Proposition 4.29, we obtain the conclusion. □

4.5 Regularity of sliding almost minimal sets II

In the previous section, we get some regularity for sliding almost minimal sets with whose boundary is a plane. In this section we will give a similar result, but with where the boundary is a C^1 manifold.

Let $\Sigma \subset \mathbb{R}^3$ be a connected closed set such that the boundary $\partial\Sigma$ is a 2-dimensional C^1 manifold. For any $x \in \partial\Sigma$, the tangent cone of Σ at x is a half space, and the boundary of the half space is the tangent plane of $\partial\Sigma$ at x.

Theorem 4.31. *Let Σ be as above, $x \in \partial\Sigma$, U be a neighborhood of x. Suppose that $E \subset \Sigma$ is an (U, h)-sliding-almost-minimal set with sliding boundary $\partial\Sigma$ and that $E \supset \partial\Sigma$. Then for each small $\tau > 0$, we can find a radius $\rho > 0$, a sliding minimal cone Z in Ω with sliding boundary L_1 and a biHölder map*

$\phi : B(x, 3\rho/2) \cap \Omega \to B(x, 2\rho) \cap \Sigma$ such that

$$
\begin{aligned}
&\phi(y) \in \partial\Sigma \text{ for } y \in L_1 \cap B(x, 3\rho/2), \ \|\phi - \mathrm{id}\|_\infty \leq 3\tau\rho,\\
&C\,|x-y|^{1+\tau} \leq |\phi(x) - \phi(y)| \leq C^{-1}\,|x-y|^{\frac{1}{1+\tau}},\\
&B(x,\rho) \cap \Sigma \subset \phi\left(B\left(x, \frac{3\rho}{2}\right) \cap \Omega\right) \subset B(x, 2\rho) \cap \Sigma,\\
&E \cap B(x,\rho) \subset \phi\left(Z \cap B\left(x, \frac{3\rho}{2}\right)\right) \subset E \cap B(x, 2\rho),
\end{aligned}
\tag{4.5.1}
$$

where Ω is the tangent cone of Σ at x and L_1 is the boundary of Ω.

The strategy of the proof will be the same as for Corollary 4.30. We do not want repeat the whole section above, because most of the statements and proofs still work. We only give a sketch.

Firstly, Lemma 4.19 is still true when we replace Ω and L_1 by Σ and $\partial\Sigma$ respectively. That is, it can be stated as follows:

Lemma 4.32. *Let Σ and $\partial\Sigma$ be as in Theorem 4.31. Suppose that $E \subset \Sigma$ is (U, h)-sliding-almost-minimal. If for each $\tau > 0$, we can find $\varepsilon_1(\tau) > 0$ such that if $x \in E \cap \partial\Sigma$ and $r > 0$ are such that*

$$B(x,r) \subset U, h(2r) \leq \varepsilon_1(\tau), \int_0^{2r} \frac{h(t)dt}{t} \leq \varepsilon_1(\tau), \theta_E(x,r) \leq \theta_E(x) + \varepsilon_1(\tau),$$

then for every $\rho \in (0, 9r/10]$ there is a sliding minimal cone Z_x^ρ such that

$$d_{x,\rho}(E, Z_x^\rho) \leq \tau,$$

and for any ball $B(y,t) \subset B(x,\rho)$,

$$|\mathcal{H}^2(E \cap B(y,t)) - \mathcal{H}^2(Z_x^\rho \cap B(y,t))| \leq \tau\rho^2,$$

where Z_x^ρ is a sliding minimal cone in Σ_x with sliding boundary T_x, where we denote by Σ_x and T_x the tangent cone of Σ at x and tangent plane of $\partial\Sigma$ at x respectively. If $E \supset \partial\Sigma$, then we can suppose that $Z_x^\rho \supset T_x$.

For each $x \in U \cap \partial\Sigma \cap E$, we see that any blow-up limit Z of E at x is a sliding minimal cone in Σ_x with sliding boundary T_x, see [11, Theorem 24.13]. If $E \supset \partial\Sigma$, we have that $Z \supset T_x$, thus $Z = T_x$ or $Z = T_x \cup Z'$, where Z' is a sliding minimal cone in Σ_x with sliding boundary T_x of type $\mathbb{P}_+$ or $\mathbb{Y}_+$. Hence, we get that $\theta_E(x) = 1$, $\frac{3}{2}$ or $\frac{7}{4}$.

Similar to Lemma 4.20, we can get that if $E \supset \partial\Sigma$ is sliding almost minimal and a blow-up limit of E at $x \in \partial\Sigma$ is the tangent plane T_x of $\partial\Sigma$ at that point, then there exists $r > 0$ such that $E \cap B(x,r) = \partial\Sigma \cap B(x,r)$. Once we get that, we can show a similar result to Lemma 4.21. That is, if $E \subset \Sigma$ is (U,h)-sliding-almost-minimal and $E \supset \partial\Sigma$, then, by putting $F = \overline{E \setminus \Sigma}$, we shall have $\mathcal{H}^2(F \cap \partial\Sigma \cap U) = 0$ and F is also (U,h)-sliding-almost-minimal. However, there is not much difference between the proof of these two facts and Lemma 4.20 and Lemma 4.21. From the later, we get that $\theta_F(x)$ can only take two values. That is, $\frac{1}{2}$ and $\frac{3}{4}$.

Finally, if $E \subset \Sigma$ is sliding almost minimal, $x \in E \cap \partial\Sigma$, we can get that if $\theta_E(x) = \frac{1}{2}$ or $\frac{3}{4}$, then the sliding minimal cone Z_x^ρ which is taken in Lemma 4.32 is of type $\mathbb{P}_+$ or $\mathbb{Y}_+$; if $E \supset \partial\Sigma$ and $\theta_E(x) = 1$, then $Z_x^\rho = T_x$; if $\theta_E(x) = \frac{3}{2}$ or $\frac{7}{4}$, then $Z_x^\rho = T_x \cup Z$, Z is of type $\mathbb{P}_+$ or $\mathbb{Y}_+$.

We also need a lemma like Lemma 4.22.

Lemma 4.33. *Let Σ and $\partial\Sigma$ be as in Lemma 4.32, U be an open set. Let $E \subset \Sigma$ be a (U,h)-sliding-almost-minimal set. Let $\varepsilon \in (0, 1/100)$ be a small number. Suppose that $x \in U \cap \partial\Sigma \cap E$, $\theta_E(x) = \frac{1}{2}$ or $\frac{3}{4}$. If $B(x, 2r_0) \subset U$, and for any $y \in E \cap \partial\Sigma \cap B(x, r_0)$ and any $0 < \rho \leq r_0$, there exists a sliding minimal cone Z_y^ρ in Ω_y (the tangent cone of Σ at y) with sliding boundary $\partial\Omega_y$ such that*

$$d_{y,r}(E, Z_y^\rho) \leq \varepsilon,$$

then there exists a radius $r > 0$ such that for any $z \in E \cap B(x,r)$, we can find a point $a \in E \cap B(x, 2r) \cap \partial\Sigma$ satisfying

$$\mathrm{dist}(z, \partial\Sigma) \geq (1 - 10\varepsilon)|z - a|.$$

Now, we state a similar result as Lemma 4.23 and Lemma 4.26, or rather, a similar result as Remark 4.25 and Remark 4.28. The proof can be adapted from the proof of Lemma 4.23 and Lemma 4.26.

Lemma 4.34. *Let Σ and $\partial\Sigma$ be as in Lemma 4.32. Let $E \subset \Sigma$ be a (U,h)-sliding-almost-minimal set such that $\theta_E(x) \in \{\frac{1}{2}, \frac{3}{2}, \frac{3}{4}, \frac{7}{4}\}$, $x \in E \cap \partial\Sigma \cap U$. If $\tau \in (0,1)$ is a small enough number, then we can find $\varepsilon_2(\tau) > 0$ such that when*

$$B(x, 10r) \subset U, h(20r) \leq \varepsilon_2(\tau), \int_0^{20r} \frac{h(t)dt}{t} \leq \varepsilon_2(\tau), \theta_E(x, 10r) \leq \theta_E(x) + \varepsilon_2(\tau),$$

for some $r > 0$, we have that for any $y \in E \cap B(x,r)$, and any $0 < t \leq 2r$, there exists a cone or a sliding minimal cone $Z(y,t)$ satisfying

$$d_{y,t}(E, Z(y,t)) \leq \tau,$$

where $Z(y,t)$ is a cone when $0 < t < \mathrm{dist}(x, \partial\Sigma)$, $Z(y,t)$ is a sliding minimal cone centered at a point in $B(x,r) \cap \partial\Sigma$ when $\mathrm{dist}(y, \partial\Sigma) \leq t \leq 2r$.

Proof of Theorem 4.31. Without loss of generality, we assume $x = 0$. Let Ω be the tangent cone of Σ at 0, L_1 be the tangent plane of $\partial\Sigma$ at 0. Then Ω is a half space, and L_1 is its boundary. Let $\tau > 0$ and $r > 0$ be as in Lemma 4.34. Since $\partial\Sigma$ is a 2-dimensional C^1 manifold, for any $\varepsilon \in (0, \tau)$, we can find a radius $0 < R < \frac{r}{2}$ and a C^1 diffeomorphism $f : \Omega \cap B(0, R) \to \Sigma$ such that $f(0) = 0$, $Df(0) = \mathrm{id}$, $f(L_1 \cap B(0, R)) \subset \partial\Sigma$ and

$$\|Df(x) - \mathrm{id}\| \leq \varepsilon.$$

We put

$$F = f^{-1}(\Sigma \cap B(0, R)).$$

For any $x \in F$ and $0 < t \leq 2r$, by Lemma 4.34, we can find a minimal cone or a sliding minimal cone $Z(f(x), t)$ such that

$$d_{f(x),t}(E, Z(f(x), t)) \leq \tau,$$

then

$$d_{x,(1-\varepsilon)t}\left(f^{-1}(E \cap B(0, R)), Z'(x, t)\right) \leq (1 + \varepsilon)\tau,$$

where we assume that $Z(f(x), t)$ is centered at a, and denote

$$Z'(x, t) = Df^{-1}(x)\,(Z(f(x), t) - a) + f^{-1}(a).$$

We note from Lemma 4.34 that if $Z(f(x), t)$ is a sliding minimal cone, then it is centered at a point in $\partial\Sigma$. Thus $a \in B(0, R) \cap \partial\Sigma$, and $Z(f(x), t)$ is a sliding minimal cone in Σ_a with sliding boundary T_a.

Since $\|Df(x) - \mathrm{id}\| \leq \varepsilon$, we have $\|Df^{-1}(x) - \mathrm{id}\| \leq 2\varepsilon$. We take $Z''(x, t) = Z(f(x), t) - a + f^{-1}(a)$, then

$$d_{x,(1-\varepsilon)t}(Z'(x, t)), Z''(x, t)) \leq 2\varepsilon,$$

thus

$$d_{x,(1-\varepsilon)t}(F, Z''(x, t)) \leq (1 + \varepsilon)\tau + 2\varepsilon.$$

$Z''(x, t)$ is a minimal cone or a sliding minimal.

Let $\mathcal{T}_a : \mathbb{R}^3 \to \mathbb{R}^3$ be the translation which send point z to $z - a + f^{-1}(a)$. Then $Z''(x, t) = \mathcal{T}_a(Z(f(x), t))$. If $Z(f(x), t)$ is a sliding minimal cone, then $Z''(x, t)$ is a slding minimal cone in $\mathcal{T}_a(\Sigma_a)$ with sliding boundary $\mathcal{T}_a(T_a)$. We put $y = f^{-1}(a)$, then it is quite easy to see that $Df(y)$ maps Ω and L_1 to $\mathcal{T}_a(\Sigma_a)$ and $\mathcal{T}_a(T_a)$ respectively. Since $\|Df(y) - \mathrm{id}\| \leq \varepsilon$, we can find a rotation $\mathcal{R}_y$ centered at point y, which will rotate $\mathcal{T}_a(\Sigma_a)$ and $\mathcal{T}_a(T_a)$ to Ω and L_1 respectively, such that

$$d_{0,1}(\mathcal{R}_y(Z''(x, t)), Z''(x, t)) \leq 2\varepsilon,$$

then $\mathcal{R}(Z''(x,t))$ is a sliding minimal cone in Ω with sliding boundary L_1.

We take $Z(x,t) = \mathcal{R}(Z''(x,t))$ when $Z''(x,t)$ is a slding minimal cone, and take $Z(x,t) = Z''(x,t)$ when $Z''(x,t)$ is a minimal cone, then

$$d_{x,(1-\varepsilon)t}(F, Z(x,t)) \leq (1+\varepsilon)\tau + 5\varepsilon < 7\tau.$$

By Corollary 4.18, we can find a radius $r' \in (0, R/2)$, a sliding minimal cone Z in Ω with sliding boundary L_1, and a biHölder map $\varphi : B(0, 3r'/2) \cap \Omega \to B(0, 2r') \cap \Omega$ such that

$$\varphi(x) \in L_1 \text{ for } x \in L_1 \cap B(0, 3r'/2),\ \|\varphi - \mathrm{id}\|_\infty \leq \tau r',$$
$$(1+\tau)^{-1}(r')^{-\tau} |x-y|^{1+\tau} \leq |\varphi(x) - \varphi(y)| \leq (1+\tau)(r')^{\frac{\tau}{1+\tau}} |x-y|^{\frac{1}{1+\tau}},$$
$$B(0,r') \cap \Omega \subset \varphi\left(B\left(0, \frac{3r'}{2}\right) \cap \Omega\right) \subset B(0, 2r') \cap \Omega,$$
$$F \cap B(0,r') \subset \varphi\left(Z \cap B\left(0, \frac{3r'}{2}\right)\right) \subset F \cap B(0, 2r').$$

We now take $\phi = f \circ \varphi$. Then $\phi : B(0, 3r'/2) \cap \Omega \to \Sigma$ is a biHölder map, and we can easily check that (4.5.1) hold if we take $\rho = r'/2$. □

4.6 Existence of two dimensional singular minimizers

Let $\Sigma \subset \mathbb{R}^3$ be a connected closed set such that the boundary $\partial\Sigma$ is a 2-dimensional connected compact C^1 manifold. Let G be any abelian group, L be a subgroup of the Čech homology group $\check{H}_1(\partial\Sigma; G)$. We say a compact set $E \supset \partial\Sigma$ spans L if L is contained in the kernel of the homomorphism induced by the inclusion map $\partial\Sigma \to E$. We set

$$\mathscr{C} = \{E \subset \Sigma \mid E \text{ spans } L\}.$$

From paper [18], we see that there exist a set $E_0 \in \mathscr{C}$, we call it a Čech minimizer, such that

$$\mathcal{H}^2(E_0 \setminus \partial\Sigma) = \inf_{E \in \mathscr{C}} \mathcal{H}^2(E \setminus \partial\Sigma). \tag{4.6.1}$$

Let's check that E_0 is also sliding minimal with boundary $\partial\Sigma$. Let $\{\varphi_t\}_{0\leq t\leq 1}$ be any sliding-deformation in Σ. We put $F = \varphi_1(E_0)$, denote by $i : \partial\Sigma \to E_0$ and $j : \partial\Sigma \to F$ the inclusion maps. We consider the map

$$\psi : \partial\Sigma \times [0,1] \to F, \psi(x,t) = \varphi_t(x).$$

It is continuous, and $\psi(x,0) = j(x)$, $\psi(x,1) = \varphi|_{\partial\Sigma}(x)$, thus the maps $j : \partial\Sigma \to F$ and $\varphi|_{\partial\Sigma} : \partial\Sigma \to F$ are homotopy equivalent. Then $j_* = (\varphi|_{\partial\Sigma})_*$, where for any map between two topology spaces $f : X \to Y$, we denote by f_* the homomorphism $\check{H}_1(X;G) \to \check{H}_1(Y;G)$ induced by the map f. However, we know that $\varphi_1|_B = \varphi_1|_{E_0} \circ i$, thus

$$j_* = (\varphi_1|_B)_* = (\varphi_1|_{E_0})_* \circ i_*.$$

But we know that $i_*(L) = 0$, thus $j_*(L) = 0$, and $F \in \mathscr{C}$, so

$$\mathcal{H}^2(F \setminus \partial\Sigma) \geq \mathcal{H}^2(E_0 \setminus \partial\Sigma),$$

E_0 is sliding minimal.

We now consider an analogous topic, that replace Čech homology by singular homology. Since $\partial\Sigma$ is a two dimensional C^1 manifold, the singular homology groups and Čech homology groups coincide, that is, $H_1(\partial\Sigma;G) = \check{H}_1(\partial\Sigma;G)$. We say that a compact subsets $E \supset \partial\Sigma$ spans L in singular homology, if L is contained in the kernel of the homomorphism $H_1(\partial\Sigma;G) \to H_1(E;G)$ induced by the inclusion map $\partial\Sigma \to E$. We consider another collection of compact sets

$$\mathscr{S} = \{E \mid E \text{ spans } L \text{ in singular homology}\}.$$

It is quite easy to see that $\mathscr{S} \subset \mathscr{C}$, that is because there is a canonical homomorphism from singular homology group to Čech homology group $H_1(E;G) \to \check{H}_1(E;G)$, and the following diagram commutes:

$$\begin{tikzcd} H_1(\partial\Sigma;G) \arrow[r] \arrow[d, equal] & H_1(E;G) \arrow[d] \\ \check{H}_1(\partial\Sigma;G) \arrow[r] & \check{H}_1(E;G). \end{tikzcd}$$

If E spans L in singular homology, then from the above commutative diagram, we can get that E spans L in Čech homology, thus $\mathscr{S} \subset \mathscr{C}$. Our goal is to find a singular minimizer, that is, we want to find a set $E \in \mathscr{S}$, we call it a singular minimizer, such that

$$\mathcal{H}^2(E \setminus \partial\Sigma) = \inf_{F \in \mathscr{S}} \mathcal{H}^2(F \setminus \partial\Sigma).$$

Proposition 4.35. *Let Σ, G, L be as above. Then there exists a singular minimizer.*

Proof. Let E_0 be a Čech minimizer. We know, from above discussion, that E_0 is sliding minimal. Thus for any $x \in E_0$, there is a neighborhood of x where E_0 is biHölder equivalent to minimal cone and by a biHölder mapping that preserves $\partial\Sigma$. By a same argument as in [10, Section 6], we conclude that E_0 is Hölder neighborhood retract. Let's check that E_0 is a singular minimizer, It is sufficient to show that E_0 spans L in singular homology. Indeed, the canonical homomorphism $H_1(E_0; G) \to \check{H}_1(E_0; G)$ is an isomorphism since E_0 is neighborhood retract, see for example [17, 27]. Now E_0 is a Čech minimizer, E_0 spans L in Čech homology, thus E_0 spans L in singular homology, and we get the conclusion. $\square$

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